LIMITS, CONTINUITY AND DIFFERENTIABILITY

LIMITS

Let $a \in R$ and let f be a function defined on a neighbourhood $]a - \delta, a + \delta[\ : \delta > 0$, of '$a$', except possibly at a. If we allow x to take values closer and closer to 'a', the function values $f(x)$ may or may not approach a particular no. L. The function f is said to have a limit L as x tends to a if $|f(x) - L|$ can be made arbitrarily small by allowing $|x - a|$ to become sufficiently small. We then write:

$$f(x) \to L \quad \text{as} \quad x \to a$$

or equivalently, $\lim_{x \to a} f(x) = L$

Note:

1. f need not be defined at $x = a$
2. Limit, if it exists, is unique

Example: Find $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2}$

Sol.: $f(x) = \dfrac{x^2 - 4}{x - 2}$ is not defined at $x = 2$. Further, $f(x) = x + 2$ as $x \neq 2$. As x moves closer to 2, $x + 2$ moves closer to 4. We say, f has the limit 4 as $x \to 2$.

Left hand limit, right hand limit

Left hand limit :(L.H.L.)

A function f tends to a limit L as x tends to 'a' from the left if $|f(x) - L|$ can be made arbitrarily small by allowing $a - x$ (positive) to become sufficiently small. We then write,

$$f(x) \to L \quad \text{as} \quad x \to a-$$

or equivalently, $\lim_{x \to a-} f(x) = L$

Right hand limit :(R.H.L.)

A function f tends to a limit L as x tends to 'a' from the right if $|f(x) - L|$ can be made arbitrarily small by allowing $(x - a)$ (positive) to become sufficiently small. We then write

$$f(x) \to L \quad \text{as} \quad x \to a+$$

or equivalently, $\lim_{x \to a+} f(x) = L$

A function f is said to have a limit L as $x \to a$ if it has both, a left hand limit and a right hand limit and they are equal. We write,

$$\lim_{x \to a} f(x) = L$$

Example: Does $\lim_{x\to 2} f(x)$ exist, where $f(x) = [x]$

Sol.: $f(x) = 1$ for $1 < x < 2$

$f(x) = 2$ for $2 < x < 3$

$\therefore$ L.H.L. $= \lim_{x\to 2-} f(x) = 1$

R.H.L. $= \lim_{x\to 2+} f(x) = 2$

L.H.L $\neq$ R.H.L. Hence $\lim_{x\to 2} f(x)$ does not exist.

Limits at infinity:

A function f is said to tend to a limit l as $x \to \infty$ if $|f(x) - l|$ can be made arbitrarily small by taking x sufficiently large.

e.g.: $$\underset{x\to\infty}{\text{Lt}} \frac{x^2+2x+1}{x^2-2x+3} = \underset{x\to\infty}{\text{Lt}} \frac{1+\frac{2}{x}+\frac{1}{x^2}}{1-\frac{2}{x}+\frac{3}{x^2}} = 1$$

Infinite limits:

A function f is said to tend to ∞ as $x \to c$ if to any number $G > 0$, there exists *a* number $\delta > 0$ such that $f(x) > G\ \forall\ x \in\ ]c - \delta, c + \delta[\ \sim\{c\}$.

e.g.: $$\underset{x\to 0}{\text{Lt}} \frac{1}{x^2} \equiv \infty$$

Properties of limits:

Let f and g be functions such that $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$. Then

(i) $\lim_{x\to a} (f+g)(x) = l + m$

(ii) $\lim_{x\to a} (f-g)(x) = l - m$

(iii) $\lim_{x\to a} (f.g)(x) = l.m$

(iv) $\lim_{x\to a} \left(\frac{f}{g}\right)(x) = \frac{l}{m}$ (if $m \neq 0$ and $g(x)$ does not vanish in at least a small interval around a, except possibly at a)

Some important limits:

(i) $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

(ii) $\lim_{\theta \to 0} \cos\theta = 1$

(iii) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

(iv) $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1} \quad : \; n \in Q$

(v) $$\lim_{n \to \infty} a^n = \begin{cases} \infty & \text{if} \quad a > 1 \\ 1 & \text{if} \quad a = 1 \\ 0 & \text{if} \quad -1 < a < 1 \\ \text{does not exist} & \text{if} \quad a \le -1 \end{cases}$$

(vi) $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$

Some important expansions:

(i) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

(ii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

(iii) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

(iv) $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \qquad : \; -1 < x \le 1$

Sandwich theorem :

Let f, g and h be functions defined on an interval I except possibly at $c \in I$. If $f(x) \le g(x) \le h(x) \; \forall \; x \in I \sim \{c\}$ and $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = l$, then $\lim_{x \to c} g(x) = l$.

CONTINUITY

Continuity at a point of a function.

Let f be a function defined on an interval I. f is said to be continuous at a point a if

(i) $f(a)$ is defined

(ii) $\lim_{x \to a} f(x)$ exists and

(iii) $\lim_{x \to a} f(x) = f(a)$

If f is not continuous at $x = a$, it is said to be discontinuous at a.

If f is continuous at each point of the interval I then f is continuous on I.

***Example*:** Check the following function for continuity at $x = 3$.

$$f(x) = \begin{cases} 6 - 5x & \text{if} \quad 1 < x < 3 \\ x - 3 & \text{if} \quad x \geq 3 \end{cases}$$

Sol.: L.H.L. $= \underset{x \to 3-}{\text{Lt}} f(x) = \underset{x \to 3-}{\text{Lt}} (6 - 5x) = -9$

R.H.L. $= \underset{x \to 3+}{\text{Lt}} f(x) = \underset{x \to 3+}{\text{Lt}} (x - 3) = 0$

$\therefore$ $\underset{x \to 3}{\text{Lt}} f(x)$ does not exist and $\therefore$ f is not continuous at $x = 3$

Properties of continuous functions:

Let $f : I \to R$ and $g : I \to R$ be functions defined over an interval I and let f and g be continuous at $a \in I$. Then,

(i) $f + g$ is continuous at $x = a$

(ii) $f - g$ is continuous at $x = a$

(iii) $f \cdot g$ is continuous at $x = a$

(iv) $\dfrac{f}{g}$ is continuous at $x = a$ (provided $g(x)$ does not vanish in a small interval around a)

Note:

1. The function $f(x) = x$ can be easily seen to be continuous at every point of its domain. By the above properties, we conclude that every polynomial function is continuous over R.
2. Trigonometric functions are continuous at every point of their domain.

Intermediate value theorem:

Let f be a function continuous on $[a, b]$ and let k_1 and $k_2 : k_1 < k_2$ denote any two values assumed by f. Then f assumes all values in the interval $[k_1, k_2]$.

Example: Show that the equation $f(x) = 0$, where $f(x) = x^3 - \sin \pi x - \dfrac{1}{2}$, has a root in $[0, 1]$.

Sol.: $f(0) = -\dfrac{1}{2}$ and $f(1) = \dfrac{1}{2}$. Hence, f assumes, by intermediate value theorem, all values in the interval $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$. In particular, f assumes the value 0 for some $x \in \,]0, 1[$.

DIFFERENTIATION

Let f be a function defined on $[a, b]$ and let $c \in\ ]a, b[$. f is said to be differentiable at $x = c$ if $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ exists finitely. Its value is denoted by $f'(c)$.

Left hand derivative (L.H.D.):

f is said to have left hand derivative at $x = c$ if $\lim_{h \to 0-} \frac{f(c+h) - f(c)}{h}$ exists finitely. Its value is denoted by $f'(c-)$.

Right hand derivative (R.H.D.):

f is said to have right hand derivative at $x = c$ if $\lim_{h \to 0+} \frac{f(c+h) - f(c)}{h}$ exists finitely. Its value is denoted by $f'(c+)$.

Note:

f is said to be differentiable at $x = c$ if and only if both $f'(c-)$ and $f'(c+)$ exist finitely and are equal.

Derivative at end point of an interval [a, b]

f is said to be differentiable at 'a' if $\lim_{h \to 0+} \frac{f(a+h) - f(a)}{h}$ exists finitely. Its value is denoted by $f'(a+)$.

f is said to be differentiable at 'b' if $\lim_{h \to 0-} \frac{f(b+h) - f(b)}{h}$ exists finitely. Its value is denoted by $f'(b-)$

Note:

1. f is differentiable on $[a, b]$ if it is differentiable at each point of $[a, b]$.

2. If $y \equiv f(x)$ is a differentiable function, then we write $\frac{dy}{dx} \equiv f'(x)$

Example: Show that $f(x) = |x|$ is not differentiable at $x = 0$.

Sol.: R.H.D. $= \underset{x \to 0+}{Lt} \frac{f(x) - f(0)}{x - 0}$

$$= \underset{x \to 0+}{Lt} \frac{|x| - 0}{x}$$

$$= \underset{x \to 0+}{Lt} \frac{x - 0}{x} = 1$$

$$\text{L.H.D.} = \underset{x \to 0-}{Lt} \frac{f(x) - f(0)}{x - 0}$$

$$= \underset{x \to 0-}{Lt} \frac{-x - 0}{x - 0} = -1$$

$\therefore$ $f'(0)$ does not exist

Properties of differentiable functions

A. If f and g are functions differentiable at $x = c$, then

(i) $(f + g)'(c) = f'(c) + g'(c)$

(ii) $(f - g)'(c) = f'(c) - g'(c)$

(iii) $(f \, . \, g)'(c) = f'(c) \, . \, g(c) + f(c) \, . \, g'(c)$

(iv) $\left(\frac{f}{g}\right)'(c) = \frac{g(c).f'(c) - f(c).g'(c)}{(g(c))^2}$ if $g(c) \neq 0$

B. If f is differentiable at c and g is differentiable at $f(c)$, then

$(gof)'(c) = g'(f(c)) \, . \, f'(c)$

Derivatives of certain functions:

1. $\frac{d}{dx}(\sin x) = \cos x$

2. $\frac{d}{dx}(\cos x) = -\sin x$

3. $\frac{d}{dx}(\tan x) = \sec^2 x$

4. $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

5. $\frac{d}{dx}(\sec x) = \sec x \tan x$

6. $\frac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x \cot x$

7. $\frac{d}{dx}(x^n) = nx^{n-1}$

8. $\frac{d}{dx}(\log x) = \frac{1}{x}$

9. $\frac{d}{dx}(a^x) = a^x \log_e a$

10. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

11. $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

12. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

13. $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

14. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$

15. $\frac{d}{dx}(\text{cosec}^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$

Derivative of inverse function:

If $y = f(x)$ and $x = g(y)$ are inverses of each other and f is differentiable, then

$$g'(y) = \frac{1}{f'(x)} \quad : \; f'(x) \neq 0$$

Derivative of a function represented parametrically.

If the system of equations

$x = \phi(t)$ and $y = \Psi(t)$: $\alpha < t < \beta$

where ϕ and Ψ are differentiable functions and $\phi'(t) \neq 0$ defines y as a function of x then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Evaluation of indeterminate forms:

1. Indeterminate forms of the type $\frac{0}{0}, \frac{\infty}{\infty}$.

The limits, $\lim_{x\to 0}\frac{x^2}{x}$ and $\lim_{x\to 0}\frac{\sin x}{x}$ are known to exist and have values 0 and 1 respectively. However, the properties of limits are not of utility in such circumstances as their numerator and denominator both, tend to zero. Such forms are called indeterminate forms.

2. If the functions f and g are differentiable in a certain neighbourhood of the point 'a' except, may be, at the point 'a' itself and $g'(x) \neq 0$ and if

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$$

or $$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$$

then $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists. The rule is called as L'Hospital's rule. The point 'a' may be either finite or improper, viz. ∞ or $-\infty$.

3. Indeterminate forms of the type $0.\infty$ or $\infty - \infty$ are reduced to forms of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by algebraic transformations.

4. Indeterminate forms of the type 1^∞, ∞^0 or 0^0 are reduced to forms of the type $0.\infty$ by taking logarithms or by the transformation.

$$f(x)^{\phi(x)} = e^{\phi(x).\ln f(x)}$$

Example: Evaluate $\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$

Sol.: $$\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x} \qquad \left(\frac{0}{0}\text{form}\right)$$

$$= \lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1 - \cos x} \qquad \left(\frac{0}{0}\text{form}\right)$$

$$= \lim_{x\to 0} \frac{e^x - e^{-x}}{\sin x} \qquad \left(\frac{0}{0}\text{form}\right)$$

$$= \lim_{x\to 0} \frac{e^x + e^{-x}}{\cos x}$$

$$= 2$$

SOLVED EXAMPLES

Ex.1: Evaluate the following limits

(i) $\lim_{x\to a}\frac{(x+2)^{5/3}-(a+2)^{5/3}}{x-a}$

(ii) $\lim_{x\to 0}\frac{5^x-4^x}{4^x-3^x}$

(iii) $\lim_{x\to 0}\frac{e^{\sin x}-1}{x}$

Sol.: (i) $\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}$

$\therefore \lim_{x\to a}\frac{(x+2)^{5/3}-(a+2)^{5/3}}{x-a}$ [Put $x + 2 = y$; $a + 2 = b$; then, $y \to b$ as $x \to a$]

$= \lim_{y\to b}\frac{y^{5/3}-b^{5/3}}{y-b}$

$= \frac{5}{3}.b^{2/3}$

$= \frac{5}{3}.(a+2)^{2/3}$

(ii) $\lim_{x\to 0}\frac{a^x-1}{x} = \log_e a$

$\therefore \lim_{x\to 0}\frac{5^x-4^x}{4^x-3^x} = \lim_{x\to 0}\frac{\frac{5^x-1}{x}-\frac{4^x-1}{x}}{\frac{4^x-1}{x}-\frac{3^x-1}{x}}$

$= \frac{\log_e 5-\log_e 4}{\log_e 4-\log_e 3}$

(iii) $\lim_{x\to 0}\frac{e^{\sin x}-1}{x} = \lim_{x\to 0}\frac{e^{\sin x}-1}{\sin x}.\frac{\sin x}{x}$

Since $\lim_{x\to 0}\frac{\sin x}{x}=1$ and

$\lim_{x\to 0}\frac{e^{\sin x}-1}{\sin x}=\lim_{y\to 0}\frac{e^y-1}{y} = 1$ [As $x \to 0$, $\sin x \to 0$]

Hence given limit is = 1

Ex.2: Evaluate

(i) $\lim_{x\to 0}\frac{x\tan x}{1-\cos x}$

(ii) $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{\sin^{-1}x}$

Sol.: (i) $\lim_{x\to 0}\frac{x\tan x}{1-\cos x} = \lim_{x\to 0}\frac{x\sin x}{2\sin^2\frac{x}{2}}.\frac{1}{\cos x}$

$$= \lim_{x\to 0}\frac{x}{\sin\frac{x}{2}}.\frac{2\sin\frac{x}{2}}{2\sin\frac{x}{2}}.\frac{\cos\frac{x}{2}}{\cos x}$$

$$= 2\lim_{x\to 0}\frac{\frac{x}{2}}{\sin\frac{x}{2}}.\frac{\cos\frac{x}{2}}{\cos x}$$

$\lim_{x\to 0}\cos\frac{x}{2}=1$; $\lim_{x\to 0}\cos x=1$; $\lim_{x\to 0}\frac{\frac{x}{2}}{\sin\frac{x}{2}}=1$

∴ Given limit = 2

(ii) $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{\sin^{-1}x}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{x\to 0}\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}.\frac{\sqrt{1+x}-\sqrt{1-x}}{\sin^{-1}x}$$

$$= \lim_{x\to 0}\frac{2x}{\sin^{-1}x}.\frac{1}{\sqrt{1+x}+\sqrt{1-x}}$$

$\lim_{x\to 0}\sqrt{1+x}+\sqrt{1-x}=2$ and

$\lim_{x\to 0}\frac{x}{\sin^{-1}x} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$ (on putting $x = \sin\theta : \theta \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$)

∴ Given limit is equal to 1

Ex.3: Evaluate

(i) $\lim_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)$

(ii) $\lim_{x\to\infty}\frac{2\sqrt{1+x^2}-3\sqrt[3]{1+x^3}}{\sqrt[4]{1+x^4}-\sqrt[5]{1+x^4}}$

Sol.: (i) $\lim_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)$

$$= \lim_{x\to\infty}\left(\frac{\sqrt{x^2+1}-\sqrt{x^2-1}}{1}\cdot\frac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{2}{\sqrt{x^2+1}+\sqrt{x^2-1}}\right)=0$$

(ii) $\lim_{x\to\infty}\frac{2\sqrt{1+x^2}-3(1+x^3)^{1/3}}{(1+x^4)^{1/4}-(1+x^4)^{1/5}}$ (Divide numerator and denominator by x)

$$= \lim_{x\to\infty}\frac{2\left(1+\frac{1}{x^2}\right)^{1/2}-3\left(1+\frac{1}{x^3}\right)^{1/3}}{\left(1+\frac{1}{x^4}\right)^{1/4}-\left(\frac{1}{x}+\frac{1}{x^5}\right)^{1/5}}$$

$$= \frac{2-3}{1-0}=-1$$

Ex.4: Evaluate

$$\lim_{n\to\infty}\left(\cos\frac{x}{2}.\cos\frac{x}{2^2}.\cos\frac{x}{2^3}.........\cos\frac{x}{2^n}\right)$$

Sol.: $S_n = \cos\frac{x}{2}.\cos\frac{x}{2^2}.........\cos\frac{x}{2^n}$

$2^n \sin\frac{x}{2^n}\; S_n = 2^{n-1}\sin\frac{x}{2^{n-1}}\; S_{n-1}$

$= 2^{n-2}\sin\frac{x}{2^{n-2}}\; S_{n-2}$

. . . .

. . . .

$= 2\sin\frac{x}{2}\; S_1$

$= \sin x$

$\therefore \quad \lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{\sin x}{2^n\sin\frac{x}{2^n}}$

$$= \lim_{n\to\infty}\frac{\sin x}{x}\cdot\frac{\left(\frac{x}{2^n}\right)}{\sin\left(\frac{x}{2^n}\right)}$$

$$= \frac{\sin x}{x}$$

Ex.5: Find the constants a, b and c if $\lim_{x\to 0}\frac{ae^{x}-b\cos x+ce^{-x}}{x\sin x}=2$

Sol.: $\lim_{x\to 0}\frac{ae^{x}-b\cos x+ce^{-x}}{x\sin x} = \lim_{x\to 0}\frac{ae^{x}-b\cos x+ce^{-x}}{x^2}\cdot\frac{x}{\sin x}$

Since the given limit and $\lim_{x\to 0}\frac{x}{\sin x}$ exist finitely and latter limit is equal to 1,

$\therefore$ $\lim_{x\to 0}\frac{ae^{x}-b\cos x+ce^{-x}}{x^2}$ exists and is finite. Consider the limit:

$$\lim_{x\to 0}\frac{ae^{x}-b\cos x+ce^{-x}}{x^2}$$

As $x \to 0$, numerator $\to a - b + c$ while denominator $\to 0$. If $a - b + c \neq 0$, then the above limit can't exist finitely. Hence, $a - b + c = 0$ and above limit is of $\frac{0}{0}$ form.

$$\therefore \quad \lim_{x\to 0}\frac{ae^{x}-b\cos x+ce^{-x}}{x^2}$$

$$= \lim_{x\to 0}\frac{ae^{x}+b\sin x-ce^{-x}}{2x} \qquad \text{(by L'Hospital's rule)}$$

(Reasoning as above, we must have $a - c = 0$ and the above limit is of $\frac{0}{0}$ form)

$$= \lim_{x\to 0}\frac{ae^{x}+b\cos x+ce^{-x}}{2} \qquad \text{(by L'Hospital's rule)}$$

$$= \frac{a+b+c}{2}$$

$\therefore$ $a - b + c = 0$ ...(i)

$a - c = 0$ ...(ii)

$\frac{a+b+c}{2} = 2$ ...(iii)

$\therefore$ $a = c$, $b = 2a$ and by (iii), $a = 1$. Hence $c = 1$ and $b = 2$

Ex.6: Discuss the continuity of the function

$$f(x) = \begin{cases} -x & \text{if} \quad x \le 0 \\ x & \text{if} \quad 0 < x \le 1 \\ 2-x & \text{if} \quad 1 < x \le 2 \\ 1 & \text{if} \quad x > 2 \end{cases}$$

at $x = 0, 1, 2$

Sol.: (a) $f(0) = 0$

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} (-x) = 0$$

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} x = 0$$

$\therefore$ $\lim_{x \to 0} f(x)$ exists and equals $f(0)$

$\therefore$ f is continuous at $x = 0$

(b) $f(1) = 1$

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} x = 1$$

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} 2 - x = 1$$

$\therefore$ $\lim_{x \to 1} f(x)$ exists and is equal to $f(1)$

$\therefore$ f is continuous at $x = 1$

(c) $f(2) = 0$

$$\lim_{x \to 2-} f(x) = \lim_{x \to 2-} 2 - x = 0$$

$$\lim_{x \to 2+} f(x) = \lim_{x \to 2+} 1 = 1$$

$\therefore$ $\lim_{x \to 2} f(x)$ does not exist and $\therefore$ f is not continuous at $x = 2$.

Ex.7: Let $f(x) = \begin{cases} \dfrac{x^2 - 2x - 3}{x+1} & \text{if} \quad x \ne -1 \\ \lambda & \text{if} \quad x = -1 \end{cases}$. If f is continuous at $x = -1$, find λ.

Sol.: $f(-1) = \lambda$;

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{x^2 - 2x - 3}{x+1}$$

$$= \lim_{x \to -1} \frac{(x+1)(x-3)}{x+1}$$

$$= -4$$

Since f is continuous at $x = -1$, we conclude that $\lambda = -4$.

Ex.8: Discuss the continuity and discontinuity of the following functions

(i) $f(x)=\begin{cases}1, & \text{if } x \text{ is rational}\\ 0, & \text{if } x \text{ is irrational}\end{cases}$ (Dirichlet's function)

(ii) $f(x)=\begin{cases}x, & \text{if } x \text{ is rational}\\ 1-x, & \text{if } x \text{ is irrational}\end{cases}$

Sol.: (i) For any $x = a$.

L.H.L = $\lim_{x\to a-} f(x)=\lim_{h\to 0} f(a-h)$ and

R. H. L = $\lim_{x\to a+} f(x)=\lim_{h\to 0} f(a+h)$

Hence, $f(x)$ oscillates between 0 and 1 as x is rattional or irrational.

L.H.L and R. H. L do not exist.

$\Rightarrow$ $f(x)$ is discontinuous at a point $x = a$ for all values of a.

(ii) For any $x = a$

$\lim_{x\to a} f(x)=\lim_{x\to a} x=a$ (when $x \to a$ through rational values)

and $\lim_{x\to a} f(x)=\lim_{x\to a}(1-x)=1-a$ (where $x \to a$ through irrational values)

Now, $\lim_{x\to a}$ will exist only when $a = 1 - a$

$\Rightarrow$ $a=\frac{1}{2}$

Thus if $x\neq\frac{1}{2}$, then $\lim_{x\to a} f(x)$ will not exist.

Hence $f(x)$ is discontinuous when $a\neq\frac{1}{2}$

Hence $f(x)$ is continoius at $x=\frac{1}{2}$.

Ex.9: (i) If $y = (x \sin x + \cos x)\,(e^x + x^2 \log x)$, find $\frac{dy}{dx}$.

(ii) If $y = a^x \sin x \log x$, find $\frac{dy}{dx}$

(iii) Find $\frac{dy}{dx}$ if $y = \log(\tan^{-1}x)$.

Sol.: (i) $\frac{dy}{dx}$ = $(\sin x + x \cos x - \sin x)\,(e^x + x^2 \log x) + (x \sin x + \cos x)\,(e^x + 2x \log x + x^2 \frac{1}{x})$

$= x \cos x\,(e^x + x^2 \log x) + (x \sin x + \cos x)(e^x + 2x \log x + x)$.

(ii) $\dfrac{dy}{dx} = a^x \log a\,(\sin x \log x) + a^x \cos x \log x + a^x \sin x\,\dfrac{1}{x}$

$$= a^x\left(\frac{\sin x}{x} + \log x\,(\cos x + \log a\,\sin x)\right)$$

(iii) $y = \log(\tan^{-1} x)$

$$\frac{dy}{dx} = \frac{1}{\tan^{-1} x}\cdot\frac{1}{1+x^2} \quad \text{by chain rule}$$

Ex.10: If $y = (\sin x)^{\tan x} + (\cos x)^{\sec x}$, find $\dfrac{dy}{dx}$.

Sol.: Let $y = u + v$ where $u = (\sin x)^{\tan x}$ and $v = (\cos x)^{\sec x}$

$$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \text{...(i)}$$

Use log differentiation to find $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$.

$$\log u = \tan x \log \sin x \Rightarrow \frac{1}{u}\frac{du}{dx} = \tan x.\frac{\cos x}{\sin x} + \sec^2 x \log \sin x$$

$$\therefore \quad \frac{du}{dx} = (\sin x)^{\tan x}\{1 + \sec^2 x.\ \log \sin x\} \qquad \text{...(ii)}$$

Similarly, $\log v = \sec x \log \cos x$

$$\frac{1}{v}\frac{dv}{dx} = \sec x\ \tan x.\ \log \cos x + \sec x.\ \frac{-\sin x}{\cos x}$$

$$\frac{dv}{dx} = (\cos x)^{\sec x}.\ \sec x \tan x\,(\log \cos x - 1) \qquad \text{...(iii)}$$

$\therefore$ from (i), (ii) & (iii) :

$$\frac{dy}{dx} = (\sin x)^{\tan x}(1 + \sec^2 x.\log \sin x) + (\cos x)^{\sec x}.\ \sec x \tan x\,(\log \cos x - 1).$$

Ex.11: If $y = x + \tan x$, show that $\cos^2 x\,\dfrac{d^2y}{dx^2} - 2y + 2x = 0$.

Sol.: $\dfrac{dy}{dx} = 1 + \sec^2 x$;

$$\frac{d^2y}{dx^2} = 2\sec^2 x \tan x$$

$$\therefore \quad \cos^2 x\,\frac{d^2y}{dx^2} - 2y = 2\tan x - 2(x + \tan x)$$

$$= -2x$$

Hence $\cos^2 x\,\dfrac{d^2y}{dx^2} - 2y + 2x = 0.$

Ex.12: If $f(a) = 2$, $f'(a) = 1$, $g(a) = -1$, $g'(a) = 2$, find

$$\lim_{x \to a} \frac{g(x) f(a) - g(a) f(x)}{x - a}$$

Sol.: $\lim_{x \to a} \frac{g(x) f(a) - g(a) f(x)}{x - a}$

$$= \lim_{x \to a} \frac{(g(x) - g(a)) f(a) - g(a)(f(x) - f(a))}{x - a}$$

$\lim_{x \to a} \frac{g(x) - g(a)}{x - a} = g'(a) = 2$

$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a) = 1$

$\therefore$ Given limit $= 2 f(a) - g(a).1$

$= 4 + 1$

$= 5$

Ex.13: If $f(x + y) = f(x) . f(y) \; \forall \; x, y \in R$, $f(5) = 2$, $f'(0) = 3$, then find $f'(5)$

Sol.: $\lim_{h \to 0} \frac{f(5 + h) - f(5)}{h}$

$$= \lim_{h \to 0} \frac{f(5). f(h) - f(5)}{h}$$

$$= 2 \lim_{h \to 0} \frac{f(h) - 1}{h}$$ $(f(0) = f(0 + 0) = f(0)^2$. If $f(0) = 0$ then , $\forall \; x \in R, f(x) = f(x + 0) = f(x) . f(0) = 0)$

$$= 2 \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$= 2 . f'(0)$

$= 6$

Ex.14: Discuss continuity and differentiability of

$$f(x) = \begin{cases} x^2 - x + 2 & \text{if} \quad x \le 1 \\ -x^2 + x & \text{if} \quad x > 1 \end{cases}$$

Sol.: $f(x)$ is obviously continuous and differentiable for $x < 1$ and $x > 1$.

$f(1) = 2$; $\lim_{x \to 1-} f(x) = 2$; $\lim_{x \to 1+} f(x) = 0$

$\therefore$ f is discontinuous at $x = 1$ and $\therefore$ f is not differentiable at $x = 1$.

Ex.15: Discuss continuity and differentiability at $x = 0$ of the function

$$f(x) = \begin{cases} x\sin\frac{1}{x} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

Sol.: For $x \neq 0$, f is obviously continuous and differentiable

$$\lim_{x\to 0}\left| x\sin\frac{1}{x} \right| \leq \lim_{x\to 0}|x| = 0$$

$\therefore \quad \lim_{x\to 0} x\sin\frac{1}{x} = 0$

$\therefore \quad f$ is continuous at $x = 0$

$$\lim_{x\to 0}\frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0}\frac{x\sin\frac{1}{x} - 0}{x}$$

$$= \lim_{x\to 0}\sin\frac{1}{x} \text{ does not exist}$$

$\therefore \quad f'(0)$ does not exist

Ex.16: Let f be a function satisfying $f(x+y) + \sqrt{6 - f(y)} = f(x)f(y)$ and $f(h) \to 6$ as $f \to 0$

Discuss the continuity of f.

Sol.: R.H.L = $\lim_{x\to x+} f(x) \; = \lim_{x\to 0} f(x)f(h) - \sqrt{6 - f(h)}$

$f(x) = \lim_{h\to 0} f(x) - \lim_{h\to 0}\sqrt{6 - f(h)}$

$f(x).6 - 0 = 6f(x) \neq f(x) \; \neq V.F$

This shows that if $f(x) \neq 0$, then f is disontinuour at x. If $f(x) = 0$, then $f(x)$ is continuoius at x.

BASIC LEVEL ASSIGNMENT

Evaluate the following limits :

1. (i) $\lim\limits_{x \to \pi/4} \dfrac{\sin x - \cos x}{x - \dfrac{\pi}{4}}$; (ii) $\lim\limits_{x \to 1} \dfrac{1 - x}{(\cos^{-1} x)^2}$; (iii) $\lim\limits_{n \to \infty} \dfrac{1^2 + 2^2 + 3^2 + \ldots\ldots + n^2}{n^3}$

2. If $f(x) = \begin{cases} \dfrac{1 - \cos 2x}{2x^2}, & x \neq 0 \\ K, & x = 0 \end{cases}$; find K for continuity of $f(x)$ at $x = 0$.

3. Show that the function $f(x) = |\sin x + \cos x|$ is continuous at $x = \pi$.

4. Find the value of $f(0)$ so that the function

$$f(x) = \frac{1}{x} - \frac{2}{e^{2x} - 1},\ x \neq 0 \text{ is continuous at } x = 0.$$

5. Find the number of points at which the function $f(x) = \dfrac{1}{\log |x|}$ is discontinuous.

6. Differentiate the following w.r.t. x :

(i) $y = \sqrt{\log\left\{\sin\left(\dfrac{x^2}{3} - 1\right)\right\}}$; (ii) $y = \sin(2^x + \log x)$; (iii) $y = \tan^{-1}\sqrt{\dfrac{1 - \cos x}{1 + \cos x}}$

7. (i) If $f(x) = (ax + b)\sin x + (cx + d)\cos x$. Find a, b, c, d such that $f'(x) = x\cos x \,\forall \in R$.

(ii) If $y = (\sin x)^{\log x}$, find $\dfrac{dy}{dx}$

(iii) If $x^y = e^{x-y}$, then prove that $\dfrac{dy}{dx} = \dfrac{\log x}{(1 + \log x)^2}$

8. Find points of discontinuity of $f(x) = \begin{cases} \sin \pi x & , \quad x < -1 \\ x^3 & , \quad -1 \le x \le 1 \\ x - x^2 & , \quad 1 < x < 2 \\ 2 - x^2 & , \quad x \ge 2 \end{cases}$

9. If $f(x) = \dfrac{[x]}{|x|}$, $x \neq 0$. ([x] is greatest integer $\le x$), find $f'(1)$.

10. If $\lim\limits_{x \to 0} \dfrac{f(x)}{x^2} = 1$ find

(i) $\lim\limits_{x \to 0} f(x)$

(ii) $\lim\limits_{x \to 0} \dfrac{f(x)}{x}$

11. Evaluate

(i) $\lim\limits_{x \to 1^-} \dfrac{\sqrt{2}\, x(x-1)}{|x-1|}$

(ii) $\lim\limits_{\theta \to 3^+} \dfrac{[\theta]}{\theta}$

(iii) $\lim\limits_{t \to 4^-} (t - [t])$

12. Evaluate the limits (if exist) of the following functions at given points and discuss about their continuity

(i) $f(x) = \dfrac{x-1}{x - \sqrt{x}}$ at $x = 1$

(ii) $f(x) = \dfrac{5 \cos x}{4x - 2\pi}$ $at\ x = \dfrac{\pi}{2}$

(iii) $f(x) = (1 + |x|)^{1/x}$ at $x = 0$

(iv) $f(x) = \dfrac{x}{1 - 2^{|x|}}$ at $x = 0$

13. Is the function $h(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$ differentiable and continuous at $x = 0$?

14. For what values of a and b will $f(x) = \begin{cases} ax & x < 2 \\ ax^2 - bx + 3 & x \ge 2 \end{cases}$

be differentiable for all values of x ?

15. Find the value of $\lim\limits_{n \to \infty} \dfrac{2}{\pi}(n+1)\cos^{-1}\left(\dfrac{1}{n}\right) - n$.

ADVANCED LEVEL ASSIGNMENT

1. If $f(x + y) = f(x)\, f(y)$ for all real x and y, $f(x) \neq 0$ for any real x and $f'(0) = 2$, find $f(x)$.

2. If $f : R \to R$ and $f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)$ for all real x, prove that $f(2) = f(1) - f(0)$.

3. Given $f(x) = [\cos x + \sin x]$, $0 < x < 2\pi$, where $[x]$ is greatest integer $\leq x$. Find points of discontinuity of $f(x)$.

4. Discuss continuity and differentiability of $f(x) = \min\{|x|, |x - 1|, |x + 1|\}$

5. Discuss the continuity and differentiability of the function

$$f(x) = \begin{cases} \dfrac{|x|(3e^{1/|x|} + 4)}{2 - e^{1/|x|}}, & x \neq 0 \\ 0 , & x = 0 \end{cases} \quad \text{at } x = 0.$$

6. Find the values of a and b in order that

$$\lim_{x \to 0} \frac{x(1 + a\cos x) - b\sin x}{\sin^3 x} \text{ may be equal to 1.}$$

7. Let $f(x) = \cos x$ and $g(x) = \begin{cases} \min f(t); 0 \leq t \leq x & 0 \leq x \leq \pi \\ \sin x - 1 & x > \pi \end{cases}$

Discuss the continuity of $g(x)$.

8. Let $f(x) = \begin{cases} -4 & -4 \leq x < 0 \\ x^2 - 4 & 0 \leq x \leq 4 \end{cases}$

Discuss the continuity and differentiability of $g(x) = f(|x|) + |f(x)|$

9. Evaluate

(i) $\lim\limits_{x \to \frac{\pi}{2}} \cos^{-1}[\cot x]$, where $[\cdot]$ denotes greatest integer function.

(ii) $\lim_{x\to 1} \frac{\sin\{x\}}{\{x\}}$, where $\{x\}$ is the fractional part function & I is any integer.

10. Evaluate : $\lim_{x\to 0}\left[\sin^2\left(\frac{\pi}{2-ax}\right)\right]^{\sec^2\left(\frac{\pi}{2-bx}\right)}$

11. Let $f(x) = \frac{\sin^{-1}(1-\{x\}).\cos^{-1}(1-\{x\})}{\sqrt{2\{x\}}.(1-\{x\})}$, then find $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$, where $\{\}$ denotes fractional part function.

12. Let $f(x) = \begin{cases} (1+|\tan x|)^{a/|\tan x|} , & -\frac{\pi}{4} < x < 0 \\ b , & x = 0 \\ e^{\cot 2x/\cot 3x} , & 0 < x < \frac{\pi}{4} \end{cases}$

Determine a and b such that f is continuous at $x = 0$.

13. Discuss the continuity and differentiability of the function :

$$f(x) = \frac{x}{1+|x|}, |x| \geq 1$$

$$= \frac{x}{1-|x|}, |x| < 1.$$

14. If $f(x + y) = f(x) + f(y) + 2xy - 1$ for all values of x and y. If $f'(0) = \cos\alpha$, then prove that $f(x) = x^2 + x\cos\alpha + 1$ for all $x \in R$.

15. Evaluate $\lim_{x\to 1} \frac{(\log(1+x)-\log 2)(3.4^{x-1}-3x)}{\{(7+x)^{1/3}-(1+3x^{1/2})\}\sin\pi x}$.

16. Discuss the continuity of the function 'f' at $x = 0$ where

$$f(x) = \begin{cases} \frac{a^{[x]+x}}{[x]+x} & x \neq 0 \\ \log a & x = 0 \end{cases}$$

17. Let $f(x)$ and $g(x)$ be defined by

$$f(x) = [x] \quad \text{and} \quad g(x) = \begin{cases} 0 & \text{if } x \in I \\ x^2 & \text{if } x \notin I \end{cases}$$

Discuss the differentiability of *fog* and *gof*.

18. Let $f(x) = \lim\limits_{n \to \infty} \dfrac{x^2 + nx \sin^3 \pi x}{1 + n \sin^3 \pi x}$, show that $f(x)$ is continuous at $x = 0$ and $x = 1$.

19. If $y = \dfrac{ax^2}{(x-a)(x-b)(x-c)} + \dfrac{bx}{(x-b)(x-c)} + \dfrac{c}{x-c} + 1$,

prove that $\dfrac{y'}{y} = \dfrac{1}{x}\left(\dfrac{a}{a-x} + \dfrac{b}{b-x} + \dfrac{c}{c-x}\right)$ where $a, b, c \in R^+$ and $x > a, b, c$.

20. If $\lim\limits_{x \to 0}\left(1 + x + \dfrac{f(x)}{x}\right)^{1/x} = e^3$, then find the function $f(x)$ and also the coefficient of x^2

OBJECTIVE ASSIGNMENT

Choose the correct option in the following :

1. The value of $\lim_{x\to 0}\left(\dfrac{\sin^{-1}x - \tan^{-1}x}{x^3}\right)$ is equal to

(a) $\dfrac{1}{2}$ (b) $\dfrac{3}{4}$ (c) $\dfrac{4}{7}$ (d) none of these

2. If $\lim_{x\to 0}\dfrac{x(1+a\cos x)+b\sin x}{x} = 1$, then

(a) $a = b$ (b) $a + b = 0$

(c) $2a = b$ (d) none of these

3. $\lim_{x\to\infty} 2x^{3/2}\left(\sqrt{x^3+1}-\sqrt{x^3-1}\right)$

(a) 0 (b) 1 (c) 2 (d) 3

4. If $\dfrac{d^2x}{dy^2}\left(\dfrac{dy}{dx}\right)^3 + \dfrac{d^2y}{dx^2} = a$, then a is equal to

(a) 0 (b) 0 (c) 2 (d) none of these

5. $\lim_{x\to\infty}\left\{\dfrac{1}{n}\left(a_1^{\frac{1}{x}} + a_2^{\frac{1}{x}} + a_3^{\frac{1}{x}} + \dots + a_n^{\frac{1}{x}}\right)\right\}^{nx}$; $a_1, a_2, \dots\dots\dots a_n$ are positive

(a) $a_1 + a_2 + \dots\dots + a_n$ (b) $a_1a_2 \dots\dots a_n$

(c) $e^{a_1a_2\dots\dots a_n}$ (d) none of these

6. $\lim_{x\to 0}\left(\left[\dfrac{x}{\sin x}\right]\right)^{\frac{1}{x}}$; $[x]$ represents greatest integer function, is

(a) 0 (b) 1 (c) 2 (d) none of these

7. Let $f(x)=\dfrac{(4^x-1)^2}{\sin\dfrac{x}{4}\log\left(1+\dfrac{x}{3}\right)}$, $x \neq 0$. If $\lim\limits_{x\to 0}\dfrac{(4^x-1)^2}{\sin\dfrac{x}{4}\log\left(1+\dfrac{x}{3}\right)} = K(\log 4)^2$, then K is equal to

(a) 12 (b) 3 (c) 7 (d) 9

8. If P(x) is a polynomial such that $P(x^2+1)=\{P(x)\}^2+1$ and P(0) = 0 then P′(0) is equal to

(a) –1 (b) 0 (c) 1 (d) 3

9. $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{\frac{\sin x}{x-\sin x}} =$

(a) 1 (b) $\dfrac{1}{e}$ (c) e (d) none of these

10. Function $f(x)=\begin{cases}\dfrac{e^{1/x}-1}{e^{\frac{1}{x}}+1}, & x\neq 0\\ -1, & x=0\end{cases}$; is

(a) continuous at $x = 0$ (b) discontinuous at $x = 0$

(c) continuous everywhere (d) none of these

11. If $f(x)=\cos\left\{\dfrac{\pi}{2}[x]-x^3\right\}, 1<x<2$ and [.] denotes the greatest integer function, then $f'\left(\sqrt[3]{\dfrac{\pi}{2}}\right)$ is equal to

(a) 0 (b) $3(\pi/2)^{2/3}$ (c) $-3(\pi/2)^{2/3}$ (d) $-\pi^{1/3}/2$

12. For $f(x)=\begin{cases}\sin \pi x, & x<-1\\ x^3, & -1\leq x\leq 1\\ x-x^2, & 1<x<2\\ 2-x^2, & x\geq 2\end{cases}$, points of discontinuity are

(a) –1, 1, 3 (b) –1, 1, 2 (c) 0, 1, 2 (d) none of these

13. Given $f(x)=\begin{cases}2(1+\tan^2\sqrt{x})^{\frac{1}{2x}}, & x\neq 0\\ Ae^{1/2}, & x=0\end{cases}$ is continuous at $x = 0$, then $A =$

(a) $\dfrac{1}{2}$ (b) 1 (c) 2 (d) none of these

14. If $f(x) = [x] + [-x]$, $[x]$ is greatest integer function and x is an integer, then for all x, $f(x)$

(a) is continuous (b) is discontinuous (c) is differentiable (d) none of these

15. Given $f(x)=\begin{cases} \dfrac{2^{\frac{1}{x}}}{1+2^{\frac{1}{x}}} & , \quad x \neq 0 \\ 1 & , \quad x=0 \end{cases}$. At $x = 0$, $f(x)$ is

(a) continuous (b) discontinuous (c) derivative exists (d) none of these

16. If $f''(x)$ is continuous at $x = 0$ and $f''(0) = 1$, then value of $\lim\limits_{x\to 0}\left(\dfrac{2f(x)-3f(2x)+f(4x)}{x^2}\right) =$

(a) 1 (b) 2 (c) 3 (d) none of these

17. If $3f(\cos x) + 2f(\sin x) = 5x$, then $f'(\cos x)$ is equal to

(a) $-5/\cos x$ (b) $5/\cos x$ (c) $-5/\sin x$ (d) $5/\sin x$

18. If $f(x) = \sin x \sin 2x \sin 3x \ldots\ldots \sin nx$, then $f'(x)$ is

(a) $\sum\limits_{K=1}^{n} K \cos Kx \quad f(x)$

(b) $2\cos x\,(2\cos 2x)\,(3\cos 3x)\ldots\ldots(n\cos nx)$

(c) $\sum\limits_{K=1}^{n} K(\cos Kx)\,(\sin Kx)$

(d) $\sum\limits_{K=1}^{n} K \cot Kx\, f(x)$

19. Given the parametric equation $x = f(t)$, $y = g(t)$, then $\dfrac{d^2y}{dx^2}$ equals

(a) $\left(\dfrac{d^2y}{dx^2}\cdot\dfrac{dx}{dt} - \dfrac{dy}{dt}\cdot\dfrac{d^2x}{dt^2}\right)\Bigg/\left(\dfrac{dx}{dt}\right)^2$

(b) $\dfrac{d^2y}{dt^2}\Bigg/\dfrac{d^2x}{dt^2}$

(c) $\dfrac{\dfrac{dx}{dt}\cdot\dfrac{d^2y}{dt^2} - \dfrac{d^2x}{dt^2}\cdot\dfrac{dy}{dt}}{\left(\dfrac{dx}{dt}\right)^3}$

(d) none of these

20. If $y^2 = P(x)$ where $P(x)$ is a polynomial of degree 3, then $2\dfrac{d}{dx}\left(y^3\dfrac{d^2y}{dx^2}\right) =$

(a) PP' (b) P^2 (c) PP''' (d) none of these

21. If $f(x) = [x \sin \pi x]$, where $[x]$ is greatest integer function, then $f'(x)$ does not exist in $(-1, 1)$ at

(a) $-\frac{1}{2}$ (b) 0 (c) $\frac{1}{2}$ (d) none of these

22. If $g(x)$ is the inverse of $f(x)$ and $f'(x) = \frac{1}{1+x^5}$ then $g'(x)$ is equal to

(a) $\frac{1}{1+f'(x)}$ (b) $\frac{1}{1+(g(x))^5}$ (c) $1 + (g(x))^5$ (d) none of these

23. If $y = |x| + |x-2|$, then $\frac{dy}{dx}$ at $x = 2$ is

(a) 0 (b) 2 (c) does not exist (d) none of these

24. If $f(x) = e^{-|x|}$, then

(a) $Lf'(0) = -1$ (b) $Rf'(0) = 1$

(c) $f(x)$ is continuous at $x = 0$ (d) none of these

25. If $f(x) = \begin{cases} cx^2 + d, & -1 < x < 1 \\ \frac{1}{|x|}, & |x| \geq 1 \end{cases}$ is continuous and differentiable, find c and d

(a) $-\frac{1}{2}, \frac{3}{2}$ (b) $0, \frac{3}{2}$ (c) $-\frac{1}{2}, 0$ (d) $\frac{3}{2}, \frac{3}{2}$

26. $\lim_{x \to 0} \frac{\int_0^{x^2} t^2 . e^{-t^2} dt}{1 - \cos(x^3)}$ is equal to

(a) $-\frac{3}{2}$ (b) $\frac{2}{3}$ (c) $\frac{4}{3}$ (d) $\frac{1}{3}$

27. $\lim_{n \to \infty} \left(\sin \frac{\pi}{2n} . \sin \frac{2\pi}{2n} ..... \sin \frac{(n-1)\pi}{n} \right)^{1/n}$ is equal to

(a) $\frac{1}{4}$ (b) $e^{4/\pi}$ (c) $e^{2/\pi}$ (d) $e^{\pi/8}$

28. Let $f(x)$ be differentiable and $f(1) = 2$ and $f'(1) = 4$, then $\lim_{x \to 1} \left(\frac{f(x)}{f(1)} \right)^{\frac{1}{x-1}}$ is equal to

(a) 1 (b) e^2 (c) 0 (d) e^{-1}

29. $\lim_{n\to\infty}\dfrac{(1+2^5+3^5+4^5+...+n^5)}{n^8}$ is

(a) 0 (b) 1/5 (c) 1/6 (d) 1/4

30. If $f(x)$ is differentiable and $f(0) = 0$, such that $2f(x+y) + f(x-y) + 3y^2 = 3f(x) - 2xy$, then

$\lim_{x\to 1}\dfrac{f(x)-1}{x-1}$ is equal to

(a) –3 (b) 0 (c) –2 (d) 1

MORE THAN ONE CORRECT CHOICE QUESTIONS

31. Let $m, n \in I^+$ and $f(x) = \dfrac{(x-1)^{2m}}{\log_e(\cos^n(x-1))}$ for all $x \in (0, 2)$. If $g(x) = e^{-|x-1|}\ \forall \in R$ and

$\lim_{x\to 1^+} f(x) = g'(1^+)$, then

(a) $m + 2n = 5$ (b) $2m + n = 4$ (c) $m - n = 1$ (d) $2m - n = 0$

32. In which of the following case(s), the limit doesn't exist ?

(a) $\lim_{x\to 0}\dfrac{x}{\sqrt{\sec^2 x - 1}}$ (b) $\lim_{x\to 0}(\sin^3 x)^{\tan x}$ (c) $\lim_{x\to\infty}\left(\dfrac{3x^2+1}{4x^2+x}\right)^{\frac{x^2+1}{2x}}$ (d) $\lim_{x\to 0}(\ln x^2)^{2x}$

33. Let $f(x)$ be differentiable function for all $x \in R^+$ and $f(1) = 1$. If $\lim_{\alpha\to x}\dfrac{\alpha^2 f(x) - x^2 f(\alpha)}{\alpha - x} = 1$ for every $x > 0$, then:

(a) $f(2) = \dfrac{17}{6}$ (b) $f(x)$ has local minima at $x = \dfrac{(2)^{1/3}}{2}$

(c) $f(x)$ is strictly increasing for all $x \geq 2$ (d) $f''(x) > 0\ \forall x \in R^+$

34. Let $f(x) = \lim_{x\to\infty}\left(\dfrac{2x}{\pi}\cot^{-1}\left(\dfrac{x}{k^2}\right)\right)$, then

(a) $f(x)$ is increasing function for all $x \in R$.

(b) $f(x)$ differentiable for all $x \in -\{0\}$

(c) $\int_{-1}^{\infty}[f(x)]dx = 0,$ where [.] represents greatest integer function

(d) $f(|x|)$ is odd function.

35. If $\lim_{x\to 1}(1+\alpha x+\beta x^2)^{\frac{\gamma}{x-1}} = \lim_{x\to\infty}\left(\frac{x^2+4x}{x+x^2}\right)^{\frac{x^2}{x+1}}$, then

(a) $\alpha+\beta=1$ (b) $\alpha+\beta=0$ (c) $\beta\gamma=4$ (d) $\beta\gamma=3$

36. Let $f(x)$ be defined in [–2, 2] by $f(x)=\begin{cases}\max\left\{\sqrt{4-x^2},\sqrt{1+x^2}\right\}; & -2\le x\le 0\\ \min\left\{\sqrt{4-x^2},\sqrt{1+x^2}\right\}; & 0<x\le 2\end{cases}$, then

(a) $f(x)$ is continuous at $x=\pm\sqrt{\frac{3}{2}}$ but non-differentiable

(b) $f(x)$ is discontinuous at $x=\pm\sqrt{\frac{3}{2}},0$

(c) $f(x)$ is non-differentiable at $x = 0$

(d) $f(x)$ is differentiable $\forall x\in(-2,2)$

37. Let f : R → R be defined by functional relationship $f\left(\frac{x+y}{3}\right)=\frac{2+f(x)+f(y)}{3}$ and $f'(0) = 2$, then which of the following statements are correct ?

(a) $y = |f(x)|$ is continuous and non-differentiable at $x = -1$.

(b) $y = \sin(f(x))$ is differentiable for all real x.

(c) $\int_{-1}^{1}[f(x)]dx=2,$ where [.] represents the greatest integer function.

(d) $\int_{1}^{2} f([x])dx=4$

38. Let $f(x) = |\sin^{-1}(\sin x)|$ $\forall x\in R,$ then

(a) $f(x)$ is non-differentiable at $x=\frac{n\pi}{2};n\in I$

(b) Number of solutions of the equation $\frac{2}{\pi}f(x)-\log_{3\pi}x=0$ are five

(c) $\int_{0}^{\pi}[f(x)]dx=\pi-2,$ where [.] represents the greatest integer function.

(d) $y = \text{sgn}(f(x))$ is continuous $\forall x\in R$

39. Let [.] denotes the greatest integer function, and $f(x) = \dfrac{\sin \frac{\pi}{4}[x]}{[x]}$, then $f(x)$ is:

(a) continuous at $x = 2$

(b) discontinuous at $x = 2$.

(c) continuous at $x = 3/2$

(d) discontinuous at $x = 3/2$.

40. If $|c| \le \dfrac{1}{2}$ and $f(x)$ is differentiable function at $x = 0$ where $f(x) = \begin{cases} b\sin^{-1}\left(\dfrac{x+c}{2}\right) & ; \quad -\dfrac{1}{2} < x < 0 \\ 1/2 & ; \quad x = 0 \\ \dfrac{e^{ax/2}-1}{x} & ; \quad 0 < x < \dfrac{1}{2} \end{cases}$

(a) $a = 2$

(b) $64b^2 + c^2 = 4$

(c) $a = 1$

(d) $16b^2 + c^2 = 64$

MISCELLANEOUS ASSIGNMENT

Comprehension-1

Let $f : R \to R$ be a differential function satisfying $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ for all $x, y \in \mathbf{R}$. If $f'(0)$ exists and equal to –1 and $f(0) = 1$.

Then answer the following questions

1. Find $f(1)$

(a) 0 (b) 1 (c) –1 (d) none of these

2. If $x \in (-\infty, 0]$ then range of $f(x)$ will be

(a) $[0, \infty)$ (b) $[1, \infty)$ (c) $[2, \infty)$ (d) none of these

3. If $g(x) = |f(x)|$ for all $x \in \mathbf{R}$, then for $g(x)$

(a) 3 non-differential point (b) 2 non-differential point

(c) 4 non-differential point (d) none of these

4. If $g(x) = f(\sin x)$, $h(x) = |g(|x|)|$ and $I(x) = g(|x|)$, then

(a) number of non-differential points for both function $h(x)$ and $I(x)$ is different

(b) range of $h(x)$ and $I(x)$ is different

(c) solution of $I(x) + \frac{1}{2} = 0$ is infinite

(d) none of these

5. Range of $f(|x|)$ is

(a) $(-\infty, 0)$ (b) $(-\infty, 0]$ (c) $(-\infty, 1]$ (d) none of these

Comprehension-2

Let $f(x)$ is polynomial function of degree six. Consider $f(x) = a_0 + a_1x + a_2x^2 + ... a_6x^6$. Such that $f(1) = 2$, $f(-1) = 0$ and satisfying

$$\lim_{x\to 0}\left\{1 + \frac{f(x)}{x^3}\right\}^{1/x} = e^2$$

Another function

$$g(x) = \lim_{m\to\infty}\frac{x^m A(x) + B(x) + 1}{2x^m + 3x + 3} \text{ and also}$$

satisfy the condition

$$\lim_{x\to 1} g(x) = \lim_{x\to 0}\left\{1 + \frac{f(x)}{x^2}\right\}^{1/x}$$

6. Find the value of a_2 and a_3

(a) 0, 1 (b) 1, 0 (c) 0, 0 (d) none of these

7. Find $\lim_{x\to 1} f(x)$ =

(a) 2 (b) 3 (c) 4 (d) none of these

8. Find the value of $g(1)$, if $2g(1) + 2A(1) - B(1) = 1$

(a) $2e^2$ (b) e^2 (c) $4e^2$ (d) none of these

Match the following:

9.

A. $\lim_{x\to 0^+}\left[\frac{\sin x}{x}\right]$	(p) 0
B. $\lim_{x\to 0^-}\left[\frac{\sin x}{x}\right]$	(q) 1
C. $\lim_{x\to 0}\left[\frac{\sin x}{x}\right]$	(r) does not exists
D. $\lim_{x\to 1^-}[\sin(\sin^{-1} x)]$	(s) –1

10. Let $f : R \to R$ be continuous quadratic function such that $f(x) - 2f\left(\frac{x}{2}\right) + f\left(\frac{x}{4}\right) = x^2$. If $f(0) = 0$, then match the following columns (I) and (II).

Column (I)	**Column (II)**
A. Value of $f'\left(\frac{9}{8}\right)$ is equal to	(p) 0
B. Total number of points of non-differentiability for $y = \|1-\|f(x) - 2\|$ is/are	(q) 2
C. If $g(x) = min\{f(t); 0 \le t \le x\}$, where $x \in [0, 4]$, then value of $g'(3)$ is	(r) 4
D. Number of locations at which $y = \|f(x)\|$ is non-differentiable is/are	(s) 6

INTEGER TYPE QUESTIONS

11. $\lim_{x\to 1}\frac{(x-1)(x^2-1)(x^3-1)(x^4-1)}{((x-1)(x^2-1))^2}$ is equal to

12. $\lim_{x\to 0}\log_{\tan^2 x}(\tan^2 2x)$ is equal to.

13. Total number of points of discontinuity of $f(x) = [3 + 4\sin x]$, where [.] denotes the greatest integer function, in $[\pi, 2\pi]$ is equal to.

14. $f(x) = [x] + \sqrt{\{x\}}$, where [.] and {.} denotes the greatest integer function and fractional part respectively, then $f(x)$ is. continuous but non-differentiable at $x = 0$. is equal to

15. Let $f\left(\frac{x+y}{n}\right)=\frac{f(x)+f(y)}{n}$ for all $x, y, \in$ R and n is a natural number other then 2. If $f'(0) = 2$ then find the value of $f(3)$.

16. Let $f(x) = x\sin(\sin x) - \sin^2 x$ and $L=\lim_{x\to 0}\frac{f(x)}{x^n}$. If limiting value '$L$' is non-zero and finite, then value of 'n' must be equal to

17. Let $L=\lim_{x\to 0}\left\{\frac{\sin^3 x}{axe^x - b\ln(1+x)+cxe^{-x}}\right\}$. If the value of L is 3/2, then $(2b + a - c)$ is equal to.

18. Let $S_n=\left(\sum_{r=1}^{n} r\right)+2\left(\sum_{r=1}^{n-1} r\right)+3\left(\sum_{r=1}^{n-2} r\right)+\ldots+n$ and $\lim_{n\to\infty}\frac{n^4}{S_n}$ is equal to 'L', then the value of $L/3$ is equal to.

19. Let $p(x)$ be a polynomial of degree 4 having the points of extremum at $x = 1$ and $x = 2$, where $\lim_{x\to 0}\left(1+\frac{p(x)}{x^2}\right)=2.$ The value of $p(2)$ is.

20. Let $f : R^+ \to R^+$ be a differentiable function satisfying $f(xy)=\frac{f(x)}{y}+\frac{f(y)}{x}\ \forall x, y\in R^+$ also $f(1) = 0$, $f'(1) = 1$. If M be the greatest value of $f(x)$ then the value of $[M + 3]$, (where [.] denotes the greatest integer function), is equal to.

PREVIOUS YEAR QUESTIONS

IIT-JEE/JEE-ADVANCE QUESTIONS

1. $\lim_{x \to 0} \frac{x \tan 2x - 2x \tan x}{(1 - \cos 2x)^2}$ is

(a) 2 (b) –2 (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

2. $\lim_{x \to 0} \frac{\sin(\pi \cos^2 x)}{x^2}$ equals

(a) $-\pi$ (b) π (c) $\frac{\pi}{2}$ (d) 1

3. Let $f : \mathbf{R} \to \mathbf{R}$ be such that $f(1) = 3$ and $f'(1) = 6$. Then $\lim_{x \to 0} \left(\frac{f(1 + x)}{f(1)} \right)^{1/x}$ equals

(a) 1 (b) $e^{1/2}$ (c) e^2 (d) e^3

4. For $x \in R$, $\lim_{x \to \infty} \left(\frac{x - 3}{x + 2} \right)^x =$

(a) e (b) e^{-1} (c) e^{-5} (d) e^5

5. The integer n for which $\lim_{x \to 0} \frac{(\cos x - 1)(\cos x - e^x)}{x^n}$ is a finite non zero number is

(a) 1 (b) 2 (c) 3 (d) 4

6. $\lim_{x \to 1} \frac{\sqrt{1 - \cos 2(x - 1)}}{x - 1}$

(a) exists and it equals $\sqrt{2}$
(b) exists and it equals $-\sqrt{2}$
(c) does not exist because $x - 1 \to 0$
(d) does not exist because left hand limit is not equal to right hand limit

7. The function $f(x) = [x]^2 - [x^2]$ (where $[y]$ is the greatest integer less than or equal to y), is discontinuous at

(a) all integers
(b) all integers except 0 and 1
(c) all integers except 0
(d) all integers except 1

8. The left-hand derivative of $f(x) = [x] \sin(\pi x)$ at $x = k$, k an integer, is

(a) $(-1)^k (k - 1)\pi$ (b) $(-1)^{k-1} (k - 1)\pi$ (c) $(-1)^k k\pi$ (d) $(-1)^{k-1} k\pi$

9. Let $h(x) = \min \{x, x^2\}$ for every real number of x. Then

(a) h is continuous for all x
(b) h is differentiable for all x
(c) $h'(x) = 1$ for all $x > 1$
(d) h is not differentiable at two values of x.

10. Let $f : \mathbf{R} \to \mathbf{R}$ be a function defined by $f(x) = \max \{x, x^2\}$. The set of all points where $f(x)$ is not differentiable, is

(a) $\{-1, 1\}$ (b) $\{-1, 0\}$ (c) $\{0, 1\}$ (d) $\{-1, 0, 1\}$

11. The function $f(x) = (x^2 - 1)\,|x^2 - 3x + 2| + \cos(|x|)$ is not differentiable at

(a) 3 (b) 0 (c) 1 (d) 2

12. Which of the following functions is differentiable at $x = 0$?

(a) $\cos(|x|) + |x|$ (b) $\cos(|x|) - |x|$ (c) $\sin(|x|) - |x|$ (d) $\sin(|x|) + |x|$

13. Consider the following statements S and R :

S : Both $\sin x$ and $\cos x$ are decreasing functions in the integral $\left(\frac{\pi}{2}, \pi\right)$.

R : If a differentiable function decreases in an interval (a, b), then its derivative also decreases in (a, b).

Which of the following is true?

(a) Both S and R are wrong

(b) Both S and R are correct, but R is not the correct explanation for S

(c) S is correct and R is the correct explanation for S

(d) S is correct and R is wrong

14. The following functions are continuous on $(0, \pi)$

(a) $\tan x$ (b) $\int_0^x t \sin\frac{1}{t}\, dt$

(c) $\begin{cases} 1, & 0 < x \le \frac{3\pi}{4} \\ 2\sin\frac{2}{9}x, & \frac{3\pi}{4} < x < \pi \end{cases}$ (d) $\begin{cases} x\sin x, & 0 < x \le \frac{\pi}{2} \\ \frac{\pi}{2}\sin(\pi + x), & \frac{\pi}{2} < x < \pi \end{cases}$

15. If $\lim_{x \to 0} \frac{((a-n)nx - \tan x)\sin nx}{x^2} = 0$, where n is nonzero real number, then a is equal to

(a) 0 (b) $\frac{n+1}{n}$ (c) n (d) $n + \frac{1}{n}$

16. The triangle formed by the tangent to the curve $f(x) = x^2 + bx - b$ at the point (1, 1) and the co-ordinate axes, lies in the first quadrant. If its area is 2, then the value of b is

(a) –1 (b) 3 (c) –3 (d) 1

17. Value of $\underset{x \to 0}{Lim}\left[(\sin x)^{1/x} + \left(\frac{1}{x}\right)^{\sin x}\right]$ is

(a) 1 (b) 0 (c) –1 (d) 2

18. $f(x)$ and $g(x)$ are two functions such that $f''(x) = -f(x)$, $g(x) = f'(x)$. Define $F(x) = f^2(x/2) + g^2(x/2)$, F (5) = 5,then F(10) is equals to

(a) 5 (b) 0 (c) 10 (d) 15

19. If $f(x) = \min(1, x^2, x^3)$, then

(a) not differentiable at 2 points

(b) continuous everywhere but not differentiable

(c) continuous $\forall x \in R$

(d) $f'(x) > 0\ \forall\ x > 1$

20. Let $f(x)$ be differentiable on the interval $(0, \infty)$ such that $f(1) = 1$ and $\lim\limits_{t \to x} \dfrac{t^2 f(x) - x^2 f(t)}{t - x} = 1$ for each $x > 0$. Then $f(x)$ is

(a) $\dfrac{1}{3x} + \dfrac{2x^2}{3}$

(b) $-\dfrac{1}{3x} + \dfrac{4x^2}{3}$

(c) $-\dfrac{1}{x} + \dfrac{2}{x^2}$

(d) $\dfrac{1}{x}$

21. $\lim\limits_{x \to \frac{\pi}{4}} \dfrac{\displaystyle\int_2^{\sec^2 x} f(t)\, dt}{x^2 - \dfrac{\pi^2}{16}}$ equals

(a) $\dfrac{8}{\pi} f(2)$

(b) $\dfrac{2}{\pi} f(2)$

(c) $\dfrac{2}{\pi} f\left(\dfrac{1}{2}\right)$

(d) $4f(2)$

22. Let $g(x) = \dfrac{(x-1)^n}{\log \cos^m (x-1)}$; $0 < x < 2$, m and n are integers, $m \neq 0$, $n > 0$ and let p be the left hand derivative of $|x - 1|$ at $x = 1$. If $\lim\limits_{x \to 1^+} g(x) = p$, then

(a) $n = 1, m = 1$

(b) $n = 1, m = -1$

(c) $n = 2, m = 2$

(d) $n > 2, m = n$

23. Let $f(x)$ be an non-constant twice differentiable function defined on $(-\infty, \infty)$ such that $f(x) = f(1 - x)$ and $f'\left(\dfrac{1}{4}\right) = 0$. Then,

(a) $f''(x)$ vanishes at least twice on $[0, 1]$

(b) $f'\left(\dfrac{1}{2}\right) = 0$

(c) $\displaystyle\int_{-1/2}^{1/2} f\left(x + \frac{1}{2}\right) \sin x\, dx = 0$

(d) $\displaystyle\int_0^{1/2} f(t) e^{\sin \pi t}\, dt = \int_{1/2}^{1} f(1 - t) e^{\sin \pi t}\, dt$

24. Let f and g be real valued functions defined on interval $(-1, 1)$ such that $g''(x)$ is continuous, $g(0) \neq 0$, $g'(0) = 0$, $g''(0) \neq 0$ and $f(x) = g(x) \sin x$.

STATEMENT-1: $\lim\limits_{x \to 0} [g(x) \cot x - g(0) \operatorname{cosec} x] = f''(0)$.

STATEMENT-2: $f'(0) = g(0)$.

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1

(b) Statement-1 and Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1

(c) Statement-1 is True, Statement-2 is False

(d) Statement-1 is False, Statement-2 is True

25. Let $g(x) = \log f(x)$ where $f(x)$ is a twice differentiable positive function on $(0, \infty)$ such that $f(x+1) = x f(x)$. Then, for $N = 1, 2, 3, \ldots$

$$g''\left(N+\frac{1}{2}\right) - g''\left(\frac{1}{2}\right) =$$

(a) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots.+\frac{1}{(2N-1)^2}\right\}$

(b) $4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots.+\frac{1}{(2N-1)^2}\right\}$

(c) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots.+\frac{1}{(2N+1)^2}\right\}$

(d) $4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots.+\frac{1}{(2N+1)^2}\right\}$

26. Let f and g be real valued functions defined on interval $(-1, 1)$ such that $g''(x)$ is continuous, $g(0) \neq 0$, $g'(0) = 0$, $g''(0) \neq 0$ and $f(x) = g(x)\sin x$.

STATEMENT-1: $\lim_{x\to 0}[g(x)\cot x - g(0)\operatorname{cosec} x] = f''(0)$.

STATEMENT-2: $f'(0) = g(0)$.

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1

(b) Statement-1 and Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1

(c) Statement-1 is True, Statement-2 is False

(d) Statement-1 is False, Statement-2 is True

27. Let $L = \lim_{x\to 0}\dfrac{a-\sqrt{a^2-x^2}-\dfrac{x^2}{4}}{x^4}$, $a > 0$. If L is finite, then

(a) $a = 2$ (b) $a = 1$ (c) $L = \frac{1}{64}$ (d) $L = \frac{1}{32}$

28. Let $p(x)$ be a polynomial of degree 4 having extremum at $x = 1, 2$ and $\lim_{x\to 0}\left(1+\frac{p(x)}{x^2}\right) = 2$.

Then the value of $p(2)$ is

29. The value of $\lim_{x\to 0}\frac{1}{x^3}\int_0^x \frac{t\ln(1+t)}{t^4+4}\,dt$ is

(a) 0 (b) $\frac{1}{12}$ (c) $\frac{1}{24}$ (d) $\frac{1}{64}$

30. Let $f: \mathbf{R} \to \mathbf{R}$ be a function such that $f(x+y) = f(x) + f(y)$, $\forall x, y \in \mathbf{R}$. If $f(x)$ is differentiable at $x = 0$, then

(a) $f(x)$ is differentiable only in a finite interval containing zero

(b) $f(x)$ is continuous $\forall x \in \mathbf{R}$

(c) $f'(x)$ is constant $\forall x \in \mathbf{R}$

(d) $f(x)$ is differentiable except at finitely many points

31. If $\lim_{x\to 0}[1+x\ln(1+b^2)]^{\frac{1}{x}} = 2b\sin^2\theta,\ b>0$ and $\theta\in(-\pi,\pi]$, then the value of θ is

(a) $\pm\frac{\pi}{4}$ (b) $\pm\frac{\pi}{3}$ (c) $\pm\frac{\pi}{6}$ (d) $\pm\frac{\pi}{2}$

32. If $f(x)=\begin{cases} -x-\frac{\pi}{2}, & x\le -\frac{\pi}{2} \\ -\cos x, & -\frac{\pi}{2}<x\le 0 \\ x-1, & 0<x\le 1 \\ \ln x, & x>1, \end{cases}$, then

(a) $f(x)$ is continuous at $x=-\frac{\pi}{2}$ (b) $f(x)$ is not differentiable at $x = 0$

(c) $f(x)$ is differentiable at $x = 1$ (d) $f(x)$ is differentiable at $x=-\frac{3}{2}$

33. Let $y'(x) + y(x)g'(x) = g(x)g'(x)$, $y(0) = 0$, $x \in \mathbf{R}$, where $f'(x)$ denotes $\frac{d\,f(x)}{dx}$ and $g(x)$ is a given non-constant differentiable function on **R** with $g(0) = g(2) = 0$. Then the value of $y(2)$ is

34. If $\lim_{x\to\infty}\left(\frac{x^2+x+1}{x+1}-ax-b\right)=4$, then

(a) $a = 1, b = 4$ (b) $a = 1, b = –4$ (c) $a = 2, b = –3$ (d) $a = 2, b = 3$

35. For $a \in \mathbb{R}$ (the set of all real numbers), $a \neq -1$, $\lim_{x\to\infty}\frac{\left(1^a+2^a+\ldots+\mathrm{n}^a\right)}{(n+1)^{a-1}[(na+1)+(na+2)+\ldots+(na+n)]}=\frac{1}{60}$

Then a =

(a) 5 (b) 7 (c) $\frac{-15}{2}$ (d) $\frac{-17}{2}$

36. Let $f : [a, b] \to [1, \infty]$ be a continuous function and let g: R → R be defined as

$$g(x)\begin{cases} 0 & \text{if} \quad x<a \\ \int_a^x f(t)dt & \text{if} \quad a\le x\le b \\ \int_a^b f(t)dt & \text{if} \quad x>b \end{cases}$$. Then

(a) $g(x)$ is continuous but not differentiable at a

(b) $g(x)$ is differentiable on R

(c) $g(x)$ is continuous but not differentiable at b

(d) $g(x)$ is continuous and differentiable at either a or b but not both

37. The largest value of the non-negative integer a for which $\lim_{x \to 1} \left\{ \frac{-ax + \sin(x-1) + a}{x + \sin(x-1) - 1} \right\}^{\frac{1-x}{1-\sqrt{x}}} = \frac{1}{4}$ is

38. Let $f : \mathrm{R} \to \mathrm{R}$ and $g : \mathrm{R} \to \mathrm{R}$ be respectively given by $f(x) = |x| + 1$ and $g(x) = x^2 + 1$. Define $h : \mathrm{R} \to \mathrm{R}$ by $h(x) = \begin{cases} \max \ \{f(x), g(x)\} & \text{if } x \le 0 \\ \min \ \{f(x), g(x)\} & \text{if } x > 0 \end{cases}$. The number of points at which $h(x)$ is not differentiable is

Paragraph For Questions 39 and 40

Given that for each $a \in (0, 1)$, $\lim_{h \to 0^+} \int_h^{1-h} t^{-a}(1-t)^{a-1}\,dt$ exists. Let this limit be $g(a)$. In addition, it is given that the function $g(a)$ is differentiable on (0, 1).

39. The value of $g\left(\frac{1}{2}\right)$ is

(a) π (b) 2π (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

40. The value of $g'\left(\frac{1}{2}\right)$ is

(a) $\frac{\pi}{2}$ (b) π (c) $-\frac{\pi}{2}$ (d) 0

DCE QUESTIONS

1. The value of $\lim_{x \to 0} \frac{e^{ax} - e^{bx}}{x}$ is

(a) 0 (b) 1 (c) $a - b$ (d) $a + b$

2. The value of $\lim_{n \to \infty} \left[\frac{1}{1.3} + \frac{1}{3.5} + \cdots + \frac{1}{(2n+1)(2n+3)} \right]$ is

(a) 1/2 (b) 1/3 (c) 1/4 (d) 1/5

3. If $f(1) = 1$ and $f'(1) = 4$, then the value of $\lim_{x \to 1} \frac{\sqrt{f(x)} - 1}{\sqrt{x} - 1}$ is

(a) 9 (b) 4 (c) 12 (d) 1

4. $\lim_{x \to 0} \frac{(1+x)^{1/x} - e + \frac{ex}{2}}{x^2}$ is equal to

(a) $\frac{11e}{24}$ (b) $\frac{-11e}{24}$ (c) $\frac{e}{24}$ (d) none of these

5. If $f(x) = \log_x (\log x)$, then $f'(x)$ at $x = e$ is

(a) 0 (b) 1 (c) $\frac{1}{e}$ (d) $\frac{1}{2}e$

6. If $f(x) = \begin{cases} x & : & x < 0 \\ 1 & : & x = 0 \\ x^2 & : & x > 0 \end{cases}$, then $\lim_{x \to 0} f(x) =$

(a) 0 (b) 1 (c) 2 (d) does not exist

7. If $f(x) = \sqrt{\dfrac{x - \sin x}{x + \cos^2 x}}$, then $\lim_{x \to \infty} f(x)$ is

(a) 0 (b) $\propto$ (c) 1 (d) none of these

8. The function $f(x) = \begin{cases} x + 2, & 1 \le x \le 2 \\ 4\ , & x = 2 \\ 3x - 2, & x > 2 \end{cases}$ is continuous at

(a) $x = 2$ only (b) $x \le 2$ (c) $x > 2$ (d) none of these

9. $\lim_{x \to 0} \dfrac{2 \sin^2 3x}{x^2} =$

(a) 0 (b) 1 (c) 18 (d) 36

10. $\lim_{x \to 0} \dfrac{e^{1/x}}{e^{1/x+1}} =$

(a) 0 (b) 1 (c) does not exist (d) none of these

11. The function $f(x) = \begin{cases} |2x - 3| \,.[x], & x \ge 1 \\ \sin\left(\dfrac{\pi x}{2}\right), & x < 1 \end{cases}$

(a) is continuous at $x = 2$
(b) is differentiable at $x = 1$
(c) is continuous but not differentiable at $x = 1$
(d) none of these

12. The value of $f(0)$ so that the function $f(x) = \dfrac{(27 - 2x)^{1/3} - 3}{9 - 3(243 + 5x)^{1/5}}$ $(x \ne 0)$ is continuous is given by

(a) $\frac{2}{3}$ (b) 6 (c) 2 (d) 4

13. The value of $\underset{x\to\infty}{\text{Lt}} \dfrac{\sqrt{1+x^4}-(1+x^2)}{x^2}$ is equal to

(a) 0 (b) –1 (c) 2 (d) none of these.

14. Let $f(x)=\begin{cases}\dfrac{\tan x-\cot x}{x-\frac{\pi}{4}}, & x\neq\dfrac{\pi}{4}\\ a & x=\dfrac{\pi}{4}\end{cases}$

the value of a so that $f(x)$ is continuous at $x=\dfrac{\pi}{4}$, is

(a) 2 (b) 4 (c) 3 (d) 1

15. The value of $\underset{\alpha\to 0}{Lt} \dfrac{\text{cosec}^{-1}(\sec\alpha)+\cot^{-1}(\tan\alpha)+\cot^{-1}\cos(\sin^{-1}\alpha)}{\alpha}$ is

(a) 0 (b) –1 (c) –2 (d) 1

16. The value of $\lim\limits_{x\to 0}\dfrac{1+\sin x-\cos x+\log(1-x)}{x^3}$ is

(a) –1 (b) 1/2 (c) –1/2 (d) 1

17. Let $f(x) = \cos x \cos 2x \cos 4x \cos 8x \cos 16x$, then the value of $f'(\pi/4)$ is

(a) $\sqrt{2}$ (b) $-\sqrt{2}$ (c) 2 (d) –2

18. The value of $\lim\limits_{x\to 0}\dfrac{(4^x-1)^3}{\sin\dfrac{x^2}{4}\log(1+3x)}$ is

(a) $\dfrac{4}{3}(\ln 4)^2$ (b) $\dfrac{4}{3}(\ln 4)^3$ (c) $\dfrac{3}{2}(\ln 4)^2$ (d) $\dfrac{3}{2}(\ln 4)^3$

AIEEE/JEE-MAINS QUESTIONS

1. If $\lim\limits_{x\to\infty}\left(1+\dfrac{a}{x}+\dfrac{b}{x^2}\right)^{2x}=e^2$, then the values of a and b, are

(a) $a\in\mathbf{R}$, $b\in\mathbf{R}$ (b) $a = 1$ and $b = 2$ (c) $a\in\mathbf{R}$, $b = 2$ (d) $a = 1$, $b\in\mathbf{R}$

2. Let $f(x) = \dfrac{1 - \tan x}{4x - \pi}$, $x \neq \dfrac{\pi}{4}$, $x \in \left[0, \dfrac{\pi}{2}\right]$. If $f(x)$ is continuous in $\left[0, \dfrac{\pi}{2}\right]$, then $f\left(\dfrac{\pi}{4}\right)$ is

(a) 1 (b) –1 (c) $-\dfrac{1}{2}$ (d) $\dfrac{1}{2}$

3. If $x = e^{y+e^{y+\cdots \text{to } \infty}}$, $x > 0$, then $\dfrac{dy}{dx}$ is

(a) $\dfrac{x}{1+x}$ (b) $\dfrac{1+x}{x}$ (c) $\dfrac{1-x}{x}$ (d) $\dfrac{1}{x}$

4. $\lim\limits_{n \to \infty} \sum\limits_{r=1}^{n} \dfrac{1}{n} e^{\frac{r}{n}}$ is

(a) e (b) $e + 1$ (c) $1 - e$ (d) $e - 1$

5. Let α and β be the distinct roots of $ax^2 + bx + c = 0$, then $\mathop{Lim}\limits_{x \to \alpha} \dfrac{1 - \cos(ax^2 + bx + c)}{(x - \alpha)^2}$ is equal to

6. If f is a real-valued differentiable function satisfying $|f(x) - f(y)| \leq (x - y)^2$, $x, y \in R$ and $f(0) = 0$, then $f(a)$ equals

(a) 2 (b) 1 (c) –1 (d) 0

7. Suppose $f(x)$ is differentiable at $x = 1$ and $\lim\limits_{h \to 0} \dfrac{1}{h} f(1+h) = 5$, then $f'(a)$ equals

(a) 5 (b) 6 (c) 3 (d) 4

8. If $\mathop{\text{Lt}}\limits_{x \to 0} \dfrac{\log(3 + x) - \log(3 - x)}{x} = k$, then value of k is

(a) $\dfrac{-1}{3}$ (b) $\dfrac{2}{3}$ (c) $\dfrac{-2}{3}$ (d) 0

9. If $f(x) = x^n$, then the value of

$$f(1) - \frac{f'(1)}{1!} + \frac{f''(1)}{2!} - \frac{f'''(1)}{3!} + \ldots + \frac{(-1)^n f^n(1)}{n!}$$ is

(a) 2^{n-1} (b) 0 (c) 1 (d) 2^n

10. If $f(x) = \begin{cases} xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$, then $f(x)$ is

(a) continues $\forall\ x \in R$, but not differentiable at $x = 0$

(b) neither differentiable nor continuous at $x = 0$

(c) discontinuous every where

(d) continuous as well as differentiable for all x

11. $\underset{x \to \pi/2}{\text{Lt}} \dfrac{[1 - \tan x/2][1 - \sin x]}{[1 + \tan x/2][\pi - 2x]^3}$ is

(a) 0 (b) $\dfrac{1}{32}$ (c) ∞ (d) $\dfrac{1}{8}$

12. Let $f(a) = g(a) = k$ and their n^{th} derivatives $f''(a)$, $g''(a)$ exist and are not equal for some n further if $\underset{x \to a}{\text{Lt}} \dfrac{f(a)\,g(x) - f(a) - g(a)\,f(x) + g(a)}{g(x) - f(x)} = 4$, then the value of k is

(a) 2 (b) 1 (c) 0 (d) 4

13. The set of points where $f(x) = \dfrac{x}{1+|x|}$ is differentiable is

(a) $(0, \infty)$ (b) $(-\infty, 0) \cup (0, \infty)$ (c) $(-\infty, -1) \cup (-1, \infty)$ (d) $(-\infty, \infty)$

14. If $x^m . y^n = (x + y)^{m+n}$, then $\dfrac{dy}{dx}$ is

(a) $\dfrac{x}{y}$ (b) $\dfrac{y}{x}$ (c) $\dfrac{x+y}{xy}$ (d) xy

15. The function $f : \mathbf{R} \setminus \{0\} \to \mathbf{R}$ given by $f(x) = \dfrac{1}{x} - \dfrac{2}{e^{2x} - 1}$ can be made continuous at $x = 0$ by defining $f(0)$ as

16. Let $f(x) = \begin{cases} (x-1)\sin\dfrac{1}{x-1} & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$. Then which one of the following is true?

(a) f is differentiable at $x = 0$ and at $x = 1$

(b) f is differentiable at $x = 0$ but not at $x = 1$

(c) f is differentiable at $x = 1$ but not at $x = 0$

(d) f is neither differentiable at $x = 0$ nor at $x = 1$

17. Let $f(x) = x|x|$ and $g(x) = \sin x$.

Statement – 1 : *gof* is differentiable at $x = 0$ and its derivative is continuous at that point.

Statement – 2 : *gof* is twice differentiable at $x = 0$.

(a) Statement-1 and Statement-2 is true; Statement-2 is ***not*** a correct explanation for Statement-1.

(b) Statement-1 is true, Statement-2 is false.

(c) Statement-1 is false, Statement-2 is true.

(d) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

18. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function defined by

$$f(x) = \frac{1}{e^x + 2e^{-x}}$$

Statement - 1 : $f(c) = \frac{1}{3}$, for some $c \in \mathbf{R}$

Statement - 2 : $0 < f(x) \le \frac{1}{2\sqrt{2}}$, for all $x \in \mathbf{R}$

(a) Statement-1 is false, Statement-2 is true.

(b) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

(c) Statement-1 is true, Statement-2 is true; Statement-2 is ***not*** a correct explanation for Statement-1

(d) Statement-1 is true, Statement-2 is false.

19. The values of p and q for which the function $f(x) = \begin{cases} \frac{\sin(p+1)x + \sin x}{x}, & x < 0 \\ q, & x = 0 \\ \frac{\sqrt{x + x^2} - \sqrt{x}}{x^{3/2}}, & x > 0 \end{cases}$

is continuous for all x in R, are

(a) $p = \frac{1}{2}, q = -\frac{3}{2}$ (b) $p = \frac{5}{2}, q = \frac{1}{2}$ (c) $p = -\frac{3}{2}, q = \frac{1}{2}$ (d) $p = \frac{1}{2}, q = \frac{3}{2}$

20. $\lim_{x \to 2} \left(\frac{\sqrt{1 - \cos\{2(x-2)\}}}{x - 2} \right)$

(a) does not exist (b) equals $\sqrt{2}$ (c) equals $-\sqrt{2}$ (d) equals $\frac{1}{\sqrt{2}}$

21. If $f : R \to R$ is a function defined by $f(x) = |x| \cos\left(\frac{2x-1}{2}\right)\pi,$ where $[x]$ denotes the greatest integer function, then f is

(a) continuous for every real x (b) discontinuous only at $x = 0$

(c) discontinuous only at non-zero integral values of x

(d) continuous only at $x = 0$

22. $\lim_{x\to 0}\frac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}$ is equal to

(a) 1 (b) 2 (c) $-\frac{1}{4}$ (d) $\frac{1}{2}$

23. $\lim_{x\to 0}\frac{\sin(\pi\cos^2 x)}{x^2}$ is equal to

(a) $\frac{\pi}{2}$ (b) 1 (c) $-\pi$ (d) π

ANSWERS

Basic Level Assignment

1. (i) $\sqrt{2}$; (ii) $\frac{1}{2}$; (iii) $\frac{1}{3}$

2. 1

4. $f(0) = 1$

5. 3

6. (i) $\dfrac{x\cot\left(\frac{x^2}{3}-1\right)}{3\sqrt{\log\sin\left(\frac{x^2}{3}-1\right)}}$; (ii) $(\cos(2^x+\log x))(2^x\log 2+\frac{1}{x})$; (iii) $\frac{1}{2}$

7. (i) $a = 1, b = 0, c = 0, d = 1$; (ii) $(\sin x)^{\log x}\left[(\log x)\cot x+\frac{1}{x}\log\sin x\right]$

8. $x = -1, 1$

9. does not exist

10. (i) 0 ; (ii) 0

11. (i) $-\sqrt{2}$; (ii) 1 ; (iii) 1

12. (i) 2, discontinuous at $x = 1$

(ii) $-5/4$, discontinuous at $x = \pi/2$

(iii) limit does not exist

(iv) limit does not exist

13. Yes, $h(x)$ is differentiable and continuous at $x = 0$.

14. (a) $a = 3/4, b = 9/4$

15. $1-\frac{2}{\pi}$

Advanced Level Assignment

1. $f(x) = e^{2x}$

3. $x = \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{3\pi}{2}, \frac{7\pi}{4}$

4. Discontinuous at $x = \pm 1, \pm\frac{1}{2}, 0$

5. Continuous at $x = 0$ but not differentiable

6. $a = -\frac{5}{2}, b = -\frac{3}{2}$

7. $g(x)$ is continuous for all x in $[0, \infty)$

8. At $x = 0$, $g(x)$ is continuous as well as differentiable

At $x = 2$, $g(x)$ is continuous but not differentiable

9. (i) limit does not exist , (ii) limit does not exist

10. e^{-a^2/b^2}

11. $\pi/2, \frac{\pi}{2\sqrt{2}}$

12. $a = 3/2, b = e^{3/2}$

13. Continuous and Differentiable at $x = 0$; Neither at $x = 1$ nor -1

15. $\frac{9}{4\pi}\log\frac{4}{e}$

16. Not continuous

17. fog is not differentiable at integral values of x. gof is differentiable everywhere.

20. $f(x) = 2x^2 + 6x^3 + ...$ the coef of $x^2 = 2$

Objective Assignment

1.	(a)	2.	(b)	3.	(c)	4.	(a)	5.	(b)
6.	(b)	7.	(a)	8.	(c)	9	(b)	10.	(b)
11.	(a)	12.	(d)	13.	(c)	14.	(a)	15.	(b)
16.	(c)	17.	(c)	18.	(d)	19.	(c)	20.	(c)
21.	(d)	22.	(c)	23.	(c)	24.	(c)	25.	(a)
26.	(b)	27.	(a)	28.	(b)	29.	(a)	30.	(c)
31.	(a,b,d)	32.	(a,b)	33.	(a,b,c,d)	34.	(b,d)	35.	(b,d)
36.	(a,c)	37.	(a,b,d)	38.	(a,b,c)	39.	(b,c)	40.	(b,c)

Miscellaneous Assignment

1.	(a)	2.	(b)	3.	(a)	4.	(d)	5.	(c)
6.	(c)	7.	(a)	8.	(b)	9.	A-(p); B-(p); C-(p); D-(p)		
10.	A-(r); B-(s); C-(p); D-(p)			11.	(6)	12.	(1)	13.	(8)
14.	(1)	15.	(6)	16.	(6)	17.	(6)	18.	(8)
19.	(0)	20.	(3)						

Previous Year Questions

IIT-JEE/JEE-Advance

1.	(c)	2.	(b)	3.	(c)	4.	(c)	5.	(c)
6.	(d)	7.	(d)	8.	(a)	9.	(a)	10.	(d)
11.	(d)	12.	(c)	13.	(d)	14.	(b)	15.	(d)
16.	(c)	17.	(a)	18.	(a)	19.	(b,c)	20.	(a)
21.	(a)	22.	(c)	23.	(a,b, c,d)	24.	(b)	25.	(a)
26.	(b)	27.	(a,c)	28.	0	29.	(b)	30.	(b, c)
31.	(d)	32.	(a,b,c,d)	33.	(0)	34.	(b)	35.	(b,d)
36.	(a,c)	37.	(2)	38.	(3)	39.	(a)	40.	(d)

DCE Questions

1.	(c)	2.	(a)	3.	(b)	4.	(a)	5.	(c)
6.	(a)	7.	(c)	8.	(d)	9.	(c)	10.	(d)
11.	(c)	12.	(c)	13.	(c)	14.	(b)	15.	(c)
16.	(c)	17.	(a)	18.	(b)				

Mains Questions

1.	(d)	2.	(c)	3.	(c)	4.	(d)	5.	(c)
6.	(d)	7.	(a)	8.	(b)	9.	(b)	10.	(a)
11.	(b)	12.	(d)	13.	(d)	14.	(b)	15.	(b)
16.	(b)	17.	(b)	18.	(b)	19.	(c)	20.	(a)
21.	(a)	22.	(b)	23.	(d)				