SOLVED PROBLEMS (SUBJECTIVE)

1. Three point charges q, 2q and 8q are to be placed on a 9 cm long straight line. Find the positions where the charges should be placed such that the potential energy of this system is minimum. In this situation, what is the electric field at the charge q due to the other two charges ?

Solution: The maximum contribution may come from the charge 8q forming pairs with others. To reduce its effect, it should be placed at a corner and the smallest charge q in the middle. This arrangement shown in figure ensures that the charges in the strongest pair 2q, 8q are at the largest separation.

The potential energy is

$$U = \frac{q^2}{4\pi\varepsilon_0}\left[\frac{2}{x} + \frac{16}{9cm} + \frac{8}{9cm - x}\right].$$

This will be minimum if

$$A = \frac{2}{x} + \frac{8}{9cm - x} \text{ is minimum.}$$

For this, $\frac{dA}{dx} = -\frac{2}{x^2} + \frac{8}{(9cm - x)^2} = 0$ …. (i)

or, 9 cm – x = 2x or, x = 3 cm

The electric field at the position of charge q is

$$\frac{q}{4\pi\varepsilon_0}\left(\frac{2}{x^2} - \frac{8}{(9cm - x)^2}\right)$$

= 0. From (i).

2. Four point charges +8 μ C, -1 μ C, -1 μ C and +8 μ C are fixed at the points $-\sqrt{27/2}$ m, $-\sqrt{3/2}$ m, $+\sqrt{3/2}$ m and $+\sqrt{27/2}$ m respectively on the y-axis. A particle of mass 6 x 10^{-4} kg and charge + 0.1 μ C moves along the –x direction. Its speed at x = ∞ is v_0. Find the least value of v_0 for which the particle will cross the origin. Find also the kinetic energy of the particle at the origin. Assume that space is gravity free. $1/(4\pi\in_0) = 9 \times 10^9$ Nm² / C².

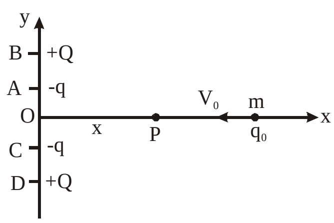
Solution : in the figure

$q = 1\,\mu C = 10^{-6}$ C,

$q_0 = +0.1\,\mu C = 10^{-7}$ C

and $m = 6 \times 10^{-4}$ kg

$Q = 8\,\mu C = 8 \times 10^{-6}$ C.



Let P be any point at a distance x from origin. Then

$AP = CP = \sqrt{3/2 + x^2}$

$BP = DP = \sqrt{27/2 + x^2}$

Electric potential at point P will be:

$$V = \frac{2KQ}{BP} - \frac{2Kq}{AP}, \text{ where}$$

$$K = \frac{1}{4\pi\varepsilon_0} = 9x10^9 \ \ Nm^2/C^2$$

$$V = 2 \times 9.0 \times 10^9 \times \left[\frac{8 \times 10^{-6}}{\sqrt{\frac{27}{2} + x^2}} - \frac{10^{-6}}{\sqrt{\frac{3}{2} + x^2}} \right] \qquad \dots (1)$$

$\therefore$ Electric field at P is :

$$E = -\frac{dV}{dx} = 1.8 \times 10^4 \times \left[8 \times \left(-\frac{1}{2}\right)\left(\frac{27}{2} + x^2\right)^{-3/2} \times 2x - 1\left(-\frac{1}{2}\right)\left(\frac{3}{2} + x^2\right) \times 2x \right]$$

E = 0 on x-axis where

$$\frac{8}{\left(\frac{27}{2} + x^2\right)^{3/2}} = \frac{1}{\left(\frac{3}{2} + x^2\right)^{3/2}}$$

$$\Rightarrow \quad \frac{(4)^{3/2}}{\left(\frac{27}{2} + x^2\right)^{3/2}} = \frac{1}{\left(\frac{3}{2} + x^2\right)^{3/2}}$$

$$\Rightarrow \quad \left(\frac{27}{2} + x^2\right) = 4\left(\frac{3}{2} + x^2\right)$$

$$\Rightarrow \quad x = \pm\sqrt{\frac{5}{2}m}$$

The least value of kinetic energy of the particles of affinity should be enough to take the particles upto $x = \pm\sqrt{\frac{5}{2}m}$ because

At $x = +\sqrt{5/2\,m}$; E = 0 F_e = 0

For x > $\sqrt{5/2\,m}$ F_e is repulsive (towards positive x-axis)

and For x < $\sqrt{5/2\,m}$; F_e is attractive (towards negative x-axis)

Now, from equation (1), potential at

$x = \sqrt{5/2\,m}$,

$$V = 1.8 \text{ x } 10^4 \left[\frac{8}{\sqrt{\frac{27}{2} + \frac{5}{2}}} - \frac{1}{\sqrt{\frac{3}{2} + \frac{5}{2}}} \right]$$

$\Rightarrow$ V = 2.7 x 10^4 volt.

Applying energy conservation at x = ∞ and x = $\sqrt{5/2}$ m,

$$\frac{1}{2}mv_0^2 = q_0V \quad \text{.... (2)}$$

$$v_0 = \sqrt{\frac{2q_0V}{m}} = \sqrt{\frac{2\times10^{-7}\times10^4}{6\times10^{-4}}}$$

$\Rightarrow$ $v_0 = 3$ m/s

$\therefore$ Minimum value of v_0 is 3 m/s.

From equation (1) potential at origin (x = 0) is

$$V_0 = 1.8 \times 10^4 \left[\frac{8}{\sqrt{27/2}} - \frac{1}{\sqrt{3/2}}\right]$$

$V_0 = 2.4 \times 10^4$ V.

Let K be the kinetic energy of the particle at origin.

Applying energy conservation at x = 0 and at x = ∞.

$$K + q_0 V_0 = \frac{1}{2}m_0v_0^2$$

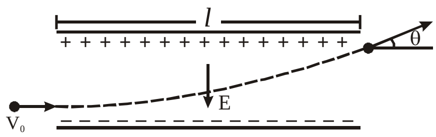
But from equation (2), $\frac{1}{2}m_0v_0^2 = q_0V$.

$\Rightarrow$ $K = q_0 (V - V_0)$

$K = 10^{-7} (2.7 \times 10^4 - 2.4 \times 10^4)$

$K = 3 \times 10^{-4}$ J.

3. A uniform electric field E is created between two parallel, charged plates as shown in figure. An electron enters the field symetrically between the plates with a speed u_0. The length of each plate is ℓ. Find the angle of deviation of the path of the electron as it comes out of the field.



Solution: The acceleration of the electron is $a = \frac{eE}{m}$ in the upward direction. The horizontal velocity remains u_0 as there is no acceleration in this direction. Thus, the time taken in crossing the field is :

$$t = \frac{l}{u_0}$$

The upward component of the velocity of the electron as it emerges from the field region is

$$u_y = at = \frac{eEl}{mu_0}$$

The horizontal component of the velocity remains

$u_x = u_0$.

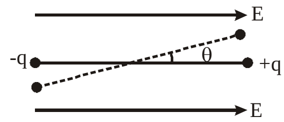
The angle θ made by the resultant velocity with the original direction is given by

$$\tan\theta = \frac{u_y}{u_x} = \frac{eEl}{mu_0^2}$$

Thus, the electron deviates by an angle

$$\theta = \tan^{-1}\frac{u_y}{u_x} = \frac{eEl}{mu_0^2}.$$

4. Figure shows an electric dipole formed by two particles fixed at the ends of a light rod of length *l*. The mass of each particle is m and the chargers are –q and +q. The system is placed in such a way that the dipole axis is parallel to a uniform electric field E that exists in the region. The dipole is slightly rotated about its centre and released. Show that for small angular displacement, the motion is angular simple harmonic and find its time period.



Solution: Suppose, the dipole axis makes an angle θ with the electric field at an instant. The magnitude of the torque on it is

$$|\vec{\tau}| = |\vec{P} \times \vec{E}|$$

$$= ql\, E \sin\theta$$

This torque will be restoring & tend to rotate the elipole back towards the electric field. Also, for small angular displacement $\sin\theta = \theta$ so that

$$\tau = -ql\, E\theta$$

It the moment of inertia of the body about OA is I, the angular acceleration becomes.

$$\alpha = \frac{\tau}{I} = -\frac{qlE}{I}\theta \qquad \alpha = -\omega^2\theta$$

where $\omega^2 = \dfrac{qlE}{I}$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{qlE}}$$

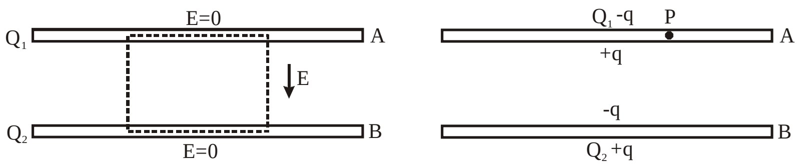
Now, moment of inertia of the system about the axis of rotation is

$$I = 2m\left(\frac{l}{2}\right)^2 - \frac{ml^2}{2}$$

So, $T = 2\pi\sqrt{\dfrac{ml}{2qE}}$.

5. Two conducting plates A and B are placed parallel to each other. A is given a charge Q_1 and B a charge Q_2. Find the distribution of charges on the four surfaces.

Solution: Consider a Gaussian surface as shown in figure. Two faces of this closed surface lie completely inside the conductor where the electric field is zero. The flux through these faces is, therefore, zero. The other parts of the closed surface which are outside the conductor are parallel to the electric field and hence the flux on these parts is also zero. The total flux of the electric field through the closed surface is, therefore, zero. From Gauss's law, the total charge inside this closed surface should be zero. The charge on the inner surface of A should be equal and opposite to the inner surface of B.



(a) (b)

The distribution should be like the one shown in figure. To find the value of q, consider the field at a point P inside the plate A. Suppose, the surface area of each plate in A. Now since $E = \frac{\sigma}{2\varepsilon_0}$ the electric field at P

Due to the charge $Q_1 - q = \frac{Q - q}{2A\varepsilon_0}$ (downward),

Due to the charge $+q = \frac{q}{2A\varepsilon_0}$ (upward),

Due to the charge $-q = \frac{q}{2A\varepsilon_0}$ (downward),

And due to the charge $Q_2 + q = \frac{Q_2 + q}{2A\varepsilon_0}$ (upward).

The net electric field at P due to all the four charged surfaces is (in the downward direction)

$$\frac{Q_1 - q}{2A\varepsilon_0} - \frac{q}{2A\varepsilon_0} + \frac{q}{A\varepsilon_0} - \frac{Q_2 + q}{2A\varepsilon_0}.$$

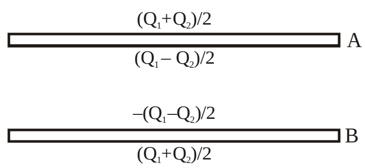
As the point P is inside the conductor, this field should be zero. Hence,

$$Q_1 - q - Q_2 - q = 0$$

or, $q = \frac{Q_1 - Q_2}{2}$ ...(i)

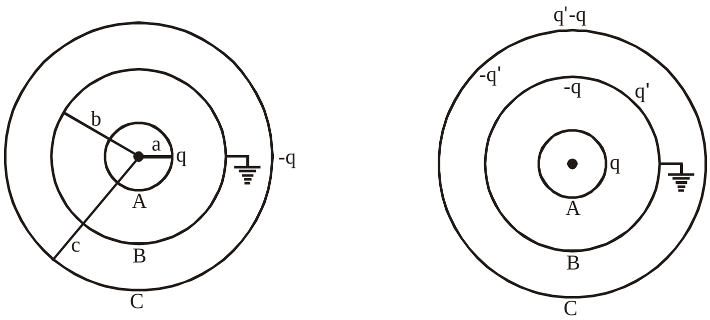
Thus, $Q_1 - q = \frac{Q_1 + Q_2}{2}$ ...(ii)

And $Q_2 + q = \frac{Q_1 + Q_2}{2}$ ...(iii)



Using these equations, the distribution shown in the figure can be redrawn as in figure.
This result is a special case of the following result. When charge conducting plate are placed parallel to each other, the two outermost surfaces get equal charges and the facing surfaces get equal and opposite charges.

6. Figure shows three concentric thin spherical shells A, B and C of radii a, b and c respectively. The shells A and C are given charges q and – q respectively and the shell B is earthed. Find the charges appearing on the surfaces of B and C.



(a) (b)

Solution: As shown in the previous worked out example, the inner surface of B must have a charge – q from the Gauss's law. Suppose, the outer a surface of B has a charge q'. The inner surface of C must have a charge $-q'$ from the Gauss's law. As the net charge on C must be – q, its outer surface should have a charge $q' - q$. The charge distribution is shown in figure. Potential of shall B due to the charge q on the surface of A.

$$= \frac{q}{4\pi\varepsilon_0 b},$$

due to the charge – q on the inner surface of B

$$= \frac{-q}{4\pi\varepsilon_0 b},$$

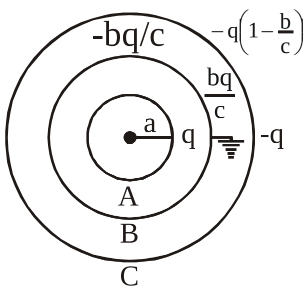
due to the charge q' on the outer surface of B

$$= \frac{q'}{4\pi\varepsilon_0 b},$$

due to the charge $-q'$, on the inner surface of C

$$= \frac{-q'}{4\pi\varepsilon_0 c}$$

and due to the charge $q' - q$ on the outer surface of C



$$= \frac{q' - q}{4\pi\varepsilon_0 c}.$$

The net potential is

$$V_B = \frac{q'}{4\pi\varepsilon_0 b} - \frac{q}{4\pi\varepsilon_0 c}.$$

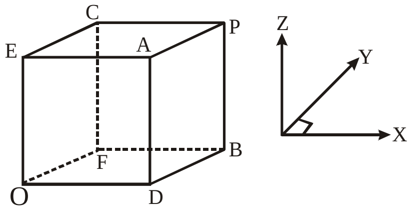
This should be zero as the shall B is earthed. Thus,

$$q' = \frac{b}{c} q.$$

The charges on various surfaces are as shown in figure

7. A cube of edge a metres carries a point charge q at each corner. Calculate the resultant force on any one of the charges.

Solution: Let us take one corner of cube as origin O(0, 0, 0) and the opposite corner as P(a, a, a). We will calculate the electric field at P due to the other seven charges at corners.



Expressing the field of a point charge in vector form

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^3}\bar{r}$$

(i) Field at P due to A, B, C

$$\bar{E}_1 = \frac{q}{4\pi\varepsilon_0 a^3}\left[\overline{AP} + \overline{BP} + \overline{CP}\right]$$

$$= \frac{q}{4\pi\varepsilon_0 a^3}\left[a\hat{j} + a\hat{k} + a\hat{i}\right]$$

(ii) Field at P due to D, E, F

Now that DP = EP = FP = $a\sqrt{2}$

$$\bar{E}_2 = \frac{q}{4\pi\varepsilon_0\left(a\sqrt{2}\right)^3}\left[\overline{DP} + \overline{EP} + \overline{FP}\right]$$

$$= \frac{q}{4\pi\varepsilon_0(2\sqrt{2}a^3)}\left[\left(a\hat{j} + a\hat{k}\right) + \left(a\hat{i} + a\hat{j}\right) + \left(a\hat{i} + a\hat{k}\right)\right]$$

$$= \frac{q}{4\pi\varepsilon_0\sqrt{2}a^2}\left[\hat{i} + \hat{j} + \hat{k}\right]$$

(iii) Field at P due to O

$$OP = a\sqrt{3}$$

$$\bar{E}_3 = \frac{q}{4\pi\varepsilon_0\left(a\sqrt{3}\right)^3}\overline{OP}$$

$$\bar{E}_3 = \frac{q}{4\pi\varepsilon_0(3\sqrt{3}a^2)}(a\hat{i} + a\hat{j} + a\hat{k})$$

$$\bar{E}_3 = \frac{q}{4\pi\varepsilon_0(3\sqrt{3}a^2)}\left(\hat{i} + \hat{j} + \hat{k}\right)$$

Resultant Field at P

$$\bar{E} = \bar{E}_1 + \bar{E}_2 + \bar{E}_3$$

$$\bar{E} = \frac{q(\hat{i} + \hat{j} + \hat{k})}{4\pi\varepsilon_0 a^2}\left[1 + \frac{1}{\sqrt{2}} + \frac{1}{3\sqrt{3}}\right]$$ outward along OP

Force on charge at P is F = q E

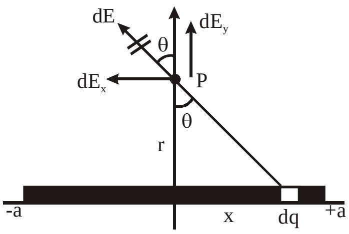
$$\Rightarrow \qquad F = \frac{q^2\sqrt{3}}{4\pi\varepsilon_0 a^2}\left[1 + \frac{1}{\sqrt{2}} + \frac{1}{3\sqrt{3}}\right]$$

Outwards along diagonal OP

Note: In this problem, we have non-coplanar point charges and hence it is best to use vector approach in general form.

8. A uniform line charge λ (in coulombs per meter) exists along the X-axis from $x = -a$ to $x = +a$. Find the electric field E at point P a distance r along the perpendicular bisector.

Solution: In all the problems which involve distributions of charge, we choose an element of charge dq to find the element of field dE' produced at the given location. The we sum all such dE's to find the total field E at that location.



You must note the symmetry of the situation. For each element dq located at positive X, there is a similar dq(see mirror-image in origin) located at the same negative value of x. The dE_x in the opposite direction due to the other dq. Hence, as we sum all the dq's along the line, all the dE_x components add to zero. So we need to sum only the dE_y components, a scalar sum since they all point in the same direction. The element of charge is $dq = \lambda\, dx$.

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2 + x^2} = \frac{\lambda\, dx}{4\pi\varepsilon_0 (r^2 + x^2)}$$

$$E_x = \int dE_x = \int dE \sin\theta = 0 \qquad \text{(by symmetry)}$$

$$E_y = \int dE_y = \int dE \cos\theta$$

$$\int \frac{\lambda dx}{4\pi\varepsilon_0 (r^2 + x^2)}\,\frac{r}{\sqrt{r^2 + x^2}} = \frac{\lambda r}{4\pi\varepsilon_0}\int \frac{dx}{(r^2 + x^2)^{3/2}}$$

The integral on the right hand side can be evaluated by substituting $x = r\tan\alpha$ and $dx = r\sec^2\alpha\, d\alpha$

$$\int \frac{dx}{(r^2 + x^2)^{3/2}} = \int \frac{r\sec^2\alpha\, d\alpha}{r^3\sec^3\alpha} = \int \frac{\cos\alpha}{r^2}d\alpha = \frac{\sin\alpha}{r^2}$$

$$\Rightarrow \qquad \int \frac{dx}{(r^2 + x^2)^{3/2}} = \frac{1}{r^2}\frac{x}{\sqrt{x^2 + r^2}}$$

$$E_y = \frac{\lambda r}{4\pi\varepsilon_0}\left|\frac{x}{r^2\sqrt{x^2 + r^2}}\right|_{-a}^{a} = \frac{\lambda r}{4\pi\varepsilon_0}\frac{2a}{r^2\sqrt{a^2 + r^2}}$$

The net field at P is E = E_y

$$E = \frac{\lambda}{2\pi\varepsilon_0 r}\frac{a}{\sqrt{a^2 + r^2}}$$

Note: In case of an infinite lien charge, the field is everywhere perpendicular to the line of charge. The field at a distance r from the line is calculated by taking $a \to \infty$ in the above result.

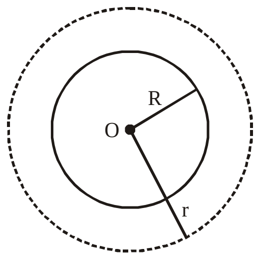
E (infinite line charge)

$$= \lim_{a\to\infty}\frac{\lambda}{2\pi\varepsilon_0 r}\frac{a}{\sqrt{a^2 + r^2}} \quad \Rightarrow \quad E = \frac{\lambda}{2\pi\varepsilon_0 r}.$$

9. A system consists of a ball of radius R carrying a spherically symmetric charge and the surrounding space filled with a charge of volume density $\rho = a/r$ where a is a constant, r is the distance from the centre of ball. Find the ball's charge at which the magnitude of the electric field is independent of r outside the ball . How high is this strength?

Solution: Let us consider a spherical surface of radius $r(r > R)$ concentric with the ball and apply Gauss's Law.

$$\oint \vec{E}.d\vec{S} == \frac{q}{\varepsilon_0}$$



Let Q= total charge of the ball

$$\varepsilon_0 E(4\pi r^2) = Q + \int_R^r \rho 4\pi x^2\,dx$$

$$\varepsilon_0 E(4\pi r^2) = Q + 4\pi\int_R^r \frac{a}{x}x^2\,dx$$

$$\varepsilon_0 E(4\pi r^2) = Q + 2\pi a(r^2 - R^2)$$

$$\Rightarrow \quad E = \left(\frac{Q - 2\pi a R^2}{4\pi\varepsilon_0}\right)\frac{1}{r^2} + \frac{2\pi a}{4\pi\varepsilon_0}$$

For E to be independent of r,

$$Q = 2\pi a R^2$$

and the value of E is

$$E = \frac{a}{2\varepsilon_0}.$$

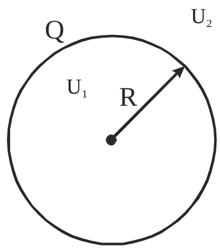
10. A charge Q is uniformly distributed over a spherical volume of radius R. Obtain an expression for the energy of the system.

Solution: In this case, the electric field exists from centre of the sphere to infinity. Potential energy is stored in electric field with energy density.

$$u = \frac{1}{2}\varepsilon_0 E^2 \text{(energy / volume)}$$

(i) Energy stored within the sphere (V_1)

Energy field at a distance r is ($r \le R$)



$$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{R^3} . r$$

$$\therefore \quad u = \frac{1}{2}\varepsilon_0 E^2$$

$$u = \frac{\varepsilon_0}{2}\left\{\frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{R^3} r\right\}^2$$

Volume of element, $dV = (4\pi r^2)dr$

$\therefore$ Energy stored in this volume, $dU = u(dV)$

$$dU = \left(4\pi r^2 dr\right)\frac{\varepsilon_0}{2}\left\{\frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{R^3} r\right\}^2$$

$$dU = \frac{1}{8\pi\varepsilon_0} \cdot \frac{Q^2}{R^6} . r^4 dr$$

$$U_1 = \int_0^R dU = \frac{1}{8\pi\varepsilon_0} \frac{Q^2}{R^6} \int_0^R f^4 dr$$

$$= \frac{Q^2}{40\pi\varepsilon_0 R^6}\left[r^5\right]_0^R$$

$$U_1 = \frac{1}{40\pi\varepsilon_0} \cdot \frac{Q^2}{R} \qquad \text{.... (1)}$$

(ii) Energy stored outside the sphere (U_2)

Electric field at a distance r is ($r \geq R$)

$$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{r^2}$$

$$u = \frac{1}{2}\varepsilon_0 E^2$$

$$u = \frac{\varepsilon_0}{2}\left\{\frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{r^2}\right\}^2$$

$dV = (4\pi r^2 dr)$

$$dU = udV = (4\pi r^2 dr)\left[\frac{\varepsilon_0}{2}\left(\frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{r^2}\right)^2\right]$$

$$dU = \frac{Q^2}{8\pi\varepsilon_0} \cdot \frac{dr}{r^2}$$

$$U_2 = \int_R^\infty dV = \frac{Q^2}{8\pi\varepsilon_0} \int_R^\infty \frac{dr}{r^2}$$

$$U_2 = \frac{Q^2}{8\pi\varepsilon_0 R} \qquad \ldots (2)$$

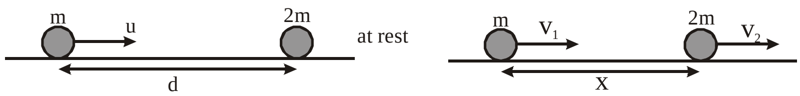
Therefore, total energy of the system is

$$U = U_1 + U_2 = \frac{Q^2}{40\pi\varepsilon_0 R} + \frac{Q^2}{8\pi\varepsilon_0 R}$$

$$U = \frac{3}{20}\frac{Q^2}{\pi\varepsilon_0 R}.$$

11. Two particles of mass m and 2m carry a charge q each. Initially the heavier particle is at rest on a smooth horizontal plane and the other is projected along the plane directly towards the first from a distance d with speed u. Find the closest distance of approach.

Solution: As the mass 2m is not fixed, it will also move away from m due to repulsion. The distance between the particles is minimum when their relative velocity is zero i.e., when they have equal velocities.



Hence at closest approach, $v_1 = v_2$

By conservation of momentum

$$mu = mv_1 + 2mv_2$$

$$v_2 = v_1 = u/3$$

By conservation of energy

Loss in KE = gain in PE

$$\frac{1}{2}mu^2 - \left(\frac{1}{2}mv_1^2 + \frac{1}{2}2mv_2^2\right) = \frac{q^2}{4\pi\varepsilon_0}\left(\frac{1}{x} - \frac{1}{d}\right)$$

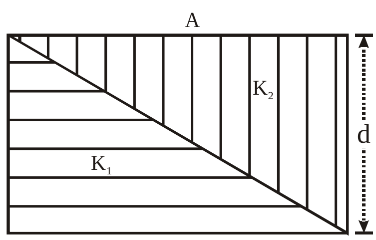
$$\frac{1}{2}mu^2 - \frac{1}{2}m\frac{u^2}{9}(1+2) = \frac{q^2}{4\pi\varepsilon_0}\left(\frac{1}{x} - \frac{1}{d}\right)$$

$$\frac{1}{3}mu^2 = \frac{q^2}{4\pi\varepsilon_0}\left(\frac{1}{x} - \frac{1}{d}\right)$$

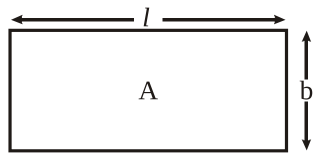
$$\frac{1}{x} = \frac{1}{d} + \frac{4\pi\varepsilon_0 mu^2}{3q^2}$$

$$x = \frac{3q^2 d}{3q^2 + 4\pi\varepsilon_0 mu^2 d}.$$

12. The capacitance of a parallel plate capacitor with plate area A and separation d is C. The space between the plates is filled with two wedges of dielectric constants K_1 and K_2 respectively (Fig.). Find the capacitance of the resulting capacitor.



Solution: Let length and breadth of the capacitor be l and b respectively and d be the distance between the plates as shown in fig. Then consider a strip at a distance x of width dx.



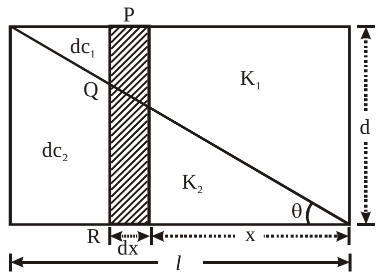
Now QR = x tan θ

and PQ = d – x tan θ

Where tan θ = d/l,

Capacitance of PQ

$$dC_1 = \frac{k_1\varepsilon_0(b\,dx)}{d - x\tan\theta} = \frac{k_1\varepsilon_0(b\,dx)}{d - \frac{xd}{l}}$$



$$dC_1 = \frac{k_1\varepsilon_0 b l dx}{d(l - x)} = \frac{k_1\varepsilon_0 A(dx)}{d(l - x)}$$

and dC_2 = capacitance of QR

$$dC_2 = \frac{k_2\varepsilon_0 b(dx)}{d\tan\theta}$$

$$dC_2 = \frac{k_2\varepsilon_0 A(dx)}{x\,d} \quad ...... \left\{\because \tan\theta = \frac{d}{l}\right\}$$

Now dC_1 and dC_2 are in series. Therefore, their resultant capacity dC will be given by

$$\frac{1}{dC} = \frac{1}{dC_1} + \frac{1}{dC_2}$$

then $$\frac{1}{dC} = \frac{1}{dC_1} + \frac{1}{dC_2}$$

$$= \frac{d(l - x)}{K_1\varepsilon_0 A(dx)} + \frac{x.d}{K_2\varepsilon_0 A(dx)}$$

$$\frac{1}{dC} = \frac{d}{\varepsilon_0 A(dx)}\left(\frac{l - x}{K_1} + \frac{x}{K_2}\right) \quad = \quad \frac{d[K_2(l - x) + K_1 x]}{\varepsilon_0 A K_1 K_2 (dx)}$$

$$dC = \frac{\varepsilon_0 A K_1 K_2}{d[K_2(l - x) + K_1 x]}dx \quad , \quad dC = \frac{\varepsilon_0 A K_1 K_2}{d[K_2 l + (K_1 - K_1)x]}dx$$

All such elemental capacitor representing DC are connected in parallel.

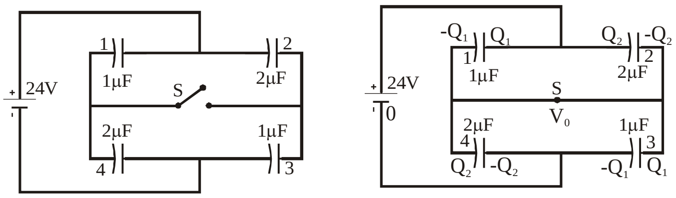
Now the capacitance of the given parallel plate capacitor is obtained by adding such infinitesimal capacitors parallel from x = 0 to x = l.

i.e. $$C = \int_{x=0}^{x=l} dC$$

$$= \int_0^l \frac{\varepsilon_0 A K_1 K_2}{d[K_2 l + (K_1 - K_2)x]}dx$$

$$C = \frac{K_1 K_2 \varepsilon_0 A}{(K_1 - K_2)d} \text{In} \frac{K_2}{K_1}.$$

13. The connections shown in figure are established with the switch S open. How much charge will flow through the switch if it is closed?



(a) (b)

Solution: When the switch is open, capacitors (2) and (3) are in series. Their equivalent capacitance is $\frac{2}{3}\mu F$. The charge appearing on each of these capacitors is, therefore, $24V \times \frac{2}{3}\mu F = 16\mu C$.

The equivalent capacitance of (1) and (4), which are also connected in series, is also $\frac{2}{3}\mu F$ and the charge on each of these capacitors is also $16\mu C$. The total charge on the two plates of (1) and (4) connected to the switch is, therefore, zero.

The situation when the switch S is closed is shown in figure. Let the charges be distributed as shown in the figure. Q_1 and Q_2 are arbitrarily chosen for the positive plates of (1) and (2). The same magnitude of charges will appear at the negative plates (3) and (4).

Take the potential at the negative terminal to the zero and at the switch to be V_0.
Writing equations for the capacitors (1), (2), (3) and (4).

$Q_1 = (24V - V_0) \times 1\ \mu F$ ...(i)

$Q_2 = (24V - V_0) \times 2\mu F$ ...(ii)

$Q_1 = V_0 \times 1\ \mu F$ ...(iii)

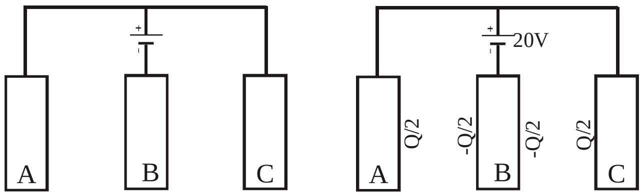
$Q_2 = V_0 \times 2\mu F$ ...(iv)

From (i) and (iii), $V_0 = 12V$.
Thus, from (iii) and (iv),

$Q_1 = 12\mu C$ and $Q_2 = 24\ \mu C$.

The charge on the two plates of (1) and (4) which are connected to the switch is, therefore $Q_2 - Q_1 = 12\mu C$. When the switch was open, this charge was zero. Thus, $12\mu C$ of charge has passed through the switch after it was closed.

14. Each of the three plates shown in figure has an area of 200 cm^2 on one side and the gap between the adjacent plates is 0.2 mm. The emf of the battery is 20 V. Find the distribution of charge on various surfaces of the plates. What is the equivalent capacitance of the system between the terminal points?



(a) (b)

Solution: Suppose the negative terminal of the battery gives a charge – Q to the plate B. As the situation is symmetric on the two sides of B, the two faces of the plate B will share equal charge –Q/2 each. From Gauss's law, the facing surfaces will have charge Q/2 each. As the positive terminal of the battery has supplied just this much charge (+Q) to A and C, the outer surfaces of A and C will have no charge. The distribution will be as shown in figure.

The capacitance between the plates A and B is

$$C = \frac{A\varepsilon_0}{d} = 8.85 \times 10^{-12}\,F/m \times \frac{200 \times 10^{-4}\,m^{-2}}{2 \times 10^{-4}\,m}$$

$$= 8.85 \times 10^{-10}\,F = 0.885\ nF\,.$$

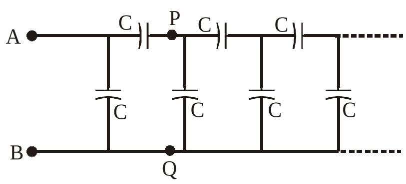
Thus, Q= 0885 nF × 20 V = 17.7 nC.

The distribution of charge on various surfaces may be written from figure

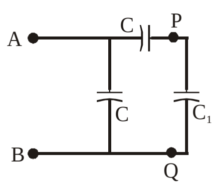
The equivalent capacitance is

$$\frac{Q}{20\,V} = 1.77\ nF\,.$$

15. Find the capacitance of the infinite ladder shown in figure.



Solution: As the ladder is infinitely long, the capacitance of the ladder to the right of the points P, Q is the same as that of the ladder to the right of the points A, B. If the equivalent capacitance of the ladder is C_1, the given ladder may be replaced by the connections shown in figure.



The equivalent capacitance between A and B is easily found to be $C + \frac{CC_1}{C + C_1}$. But being equivalent to the original ladder, the equivalent capacitance is also C_1.

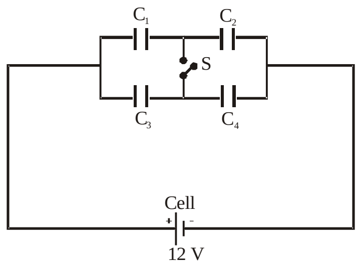
Thus, $$C_1 = C + \frac{CC_1}{C + C_1}$$

Or, $C_1C + C_1^2 = C^2 + 2CC_1$

Or, $C_1^2 - CC_1 - C^2 = 0$

Giving $C_1 = \dfrac{C + \sqrt{C^2 + 4C^2}}{2} = \dfrac{1+\sqrt{5}}{2}C$.

Negative value of C_1 is rejected.



16. The emf of the cell in the circuit is 12 volts and the capacitors are : $C_1 = 1\ \mu f, C_2 = 3\ \mu f, C_3 = 2\ \mu f, C_4 = 4\ \mu f$. Calculate the charge on each capacitor and the total charge drawn from the cell when

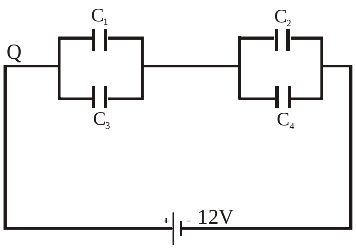
(a) the switch s is closed

(b) the switch s is open.

Solution:(a) Switch S is closed :

$$C = \frac{(C_1 + C_3)(C_2 + C_4)}{(C_1 + C_3) + (C_2 + C_4)}$$

$$C = \frac{3 \times 7}{3 + 7} = 2.1\mu F$$



total charge drawn from the cell is :

Q = C V = 2.1 μF × 12 volts = 25.2 μC

C_1, C_3 are in parallel and C_2, C_4 are in parallel.

Charge on C_1

$$Q_1 = \frac{C_1}{C_1 + C_3}Q = \frac{1}{1+2} \times 25.2\mu C = 8.4\mu C.$$

Charge on C_3

$$Q_3 = \frac{C_3}{C_1 + C_3}Q = \frac{2}{1+2} \times 25.2\mu C = 16.8\mu C.$$

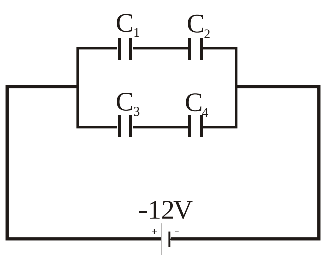
Charge on C_2

$$Q_2 = \frac{C_2}{C_2 + C_4}Q = \frac{3}{3+4} \times 25.2\mu C = 10.8\mu C.$$

Charge on C_4

$$Q_4 = \frac{C_4}{C_2 + C_4}Q = \frac{4}{3+4} \times 25.2\mu C = 14.4\mu C.$$

(b) Switch S is open :



$$C = \frac{C_1C_2}{C_1 + C_2} + \frac{C_3C_4}{C_3 + C_4}$$

$$C = \frac{1 \times 3}{1+3} + \frac{2 \times 4}{2+4} = \frac{25}{12}\mu F$$

total charge drawn from battery is :

$$Q = CV = \frac{25}{12} \times 12 = 25\mu C$$

C_1 & C_2 are in series and the potential difference across combination is 12 volts.
charge on C_1 = charge on C_2

$$= \left(\frac{C_1 C_2}{C_1 + C_2} \right) V = \frac{3}{4} \times 12 = 9\mu C.$$

C_3 & C_4 are in series and the potential difference across combination is 12 volts.
charge on C_3 = charge on C_4

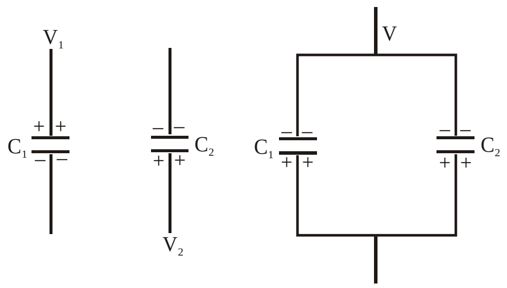
$$= \left(\frac{C_3 C_4}{C_3 + C_4} \right) V = \frac{8}{6} \times 12 = 16\mu C.$$

17. Two capacitors $C_1 = 1\mu F$ and $C_2 = 4\mu F$ are charged to a potential difference of 100 volts and 200 volts respectively. The charged capacitors are now connected to each other with terminals of opposite sign connected together. What is the

(a) final charge on each capacitor in steady state ?

(b) decrease in the energy of the system ?

Solution:



Initial charge on $C_1 = C_1 V_1 = 100\mu C$

Initial charge on $C_2 = C_2 V_2 = 800\mu C$

$$C_1 V_1 < C_2 V_2$$

when the terminals of opposite polarity are connected together, the magnitude of net charge finally is equal to the difference of magnitude of charges before connection.

(charge on $C_2)_i$ - (charge on $C_1)_i$
= (charge on $C_2)_f$ - (charge on $C_1)_f$

Let V be the final common potential difference across each.

The charges will be redistributed and the system attains a steady state when potential difference across each capacitor becomes same.

$$C_2 V_2 - C_1 V_1 = C_2 V + C_1 V$$

$$V = \frac{C_1 V_2 - C_1 V_1}{C_2 + C_1} = \frac{800 - 100}{5} = 140 \text{ volts}$$

Note that because $C_1V_1 < C_2V_2$, the final charge polarities are same as that of C_2 before connection.

Final charge on $C_1 = C_1\ V = 140\ \mu C$

Final charge on $C_2 = C_2\ V = 560\ \mu C$

Loss of energy $= U_i - U_f$,

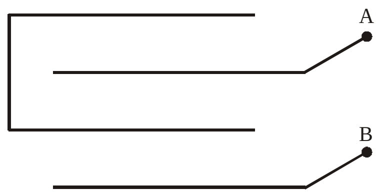
Loss of energy $= \frac{1}{2}C_1V_1^2 + \frac{1}{2}C_2V_2^2 - \frac{1}{2}C_1V^2 - \frac{1}{2}C_2V^2$

$$= \frac{1}{2}1(100)^2 + \frac{1}{2}4(200)^2 - \frac{1}{2}(1+4)(140)^2$$

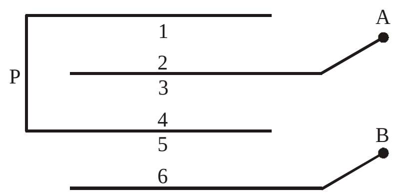
$$= 36000\mu J = 0.036J$$

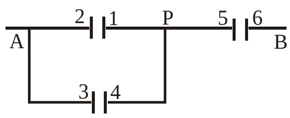
Note: The energy is lost as heat in the connected wires due to the temporary currents that flow while the charge is being redistributed.

18. Four identical metal plates are located in air at equal separations d as shown. The area of each plate is A. Calculate the effective capacitance of the arrangement across A and B.



Solution : Let us call the isolated plate as P. A capacitor is formed by a pair of parallel plates facing each other. Hence we have three capacitor formed by the pairs (1, 2), (3, 4) and (5, 6). The surface 2 and 3 are at same potential as that of A. The arrangement can be redrawn as a network of three capacitors.

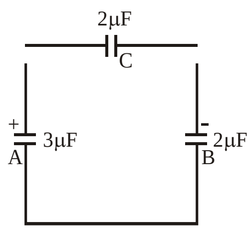




$$C_{AB} = \frac{2C.C}{2C+C} = \frac{2C}{3}$$

$$= \frac{2}{3}\frac{\varepsilon_0 A}{d}$$

19. Two capacitors A and B with capacities $3\mu F$ and $2\mu F$ are charged to a potential difference of 100V and 180V respectively. The plates of the capacitors are connected as shown in the figure with one wire of each capacitor free. The upper plate of A is positive and that of B is negative. An uncharged $2\mu F$ capacitor C with lead wires falls on the free ends to complete the circuit. Calculate :



(i) the final charge on the three capacitors, and

(ii) the amount of electrostatic energy stored in the system before and afterthe completion of the circuit.

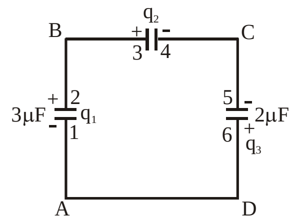
Solution(i): Charge on capacitor A, before joining with an uncharged capacitor,

$q_A = CV = (100) \times 3\ \mu c = 300\ \mu c$

similarly charge on capacitor B,

$q_B = 180 \times 2\ \mu c$

$= 360\ \mu c$



Let q_1, q_2 and q_3 be the charges on the three capacitors after joining them as shown in fig.

From conservation of charge,

Net charge on plates 2 and 3 before joining

= Net charge after joining

$\therefore$ $300 = q_1 + q_2$ .... (1)

Similarly, net charge on plates 4 and 5 before joining

= Net charge after joining

$-360 = -q_2 - q_3$

$360 = q_2 + q_3$ .... (2)

applying Kirchoff's 2nd law in loop ABCDA,

$$\frac{q_1}{3} - \frac{q_2}{2} + \frac{q_3}{2} = 0$$

$2q_1 - 3q_2 + 3q_3 = 0$ .... (3)

From equations (1), (2) and (3),

$q_1 = 90\ \mu c$, $q_2 = 90\ \mu c$ and $q_3 = 150\ \mu c$

(ii) (a) Electrostatic energy stored before completing the circuit,

$$U_i = \frac{1}{2}(3 \times 10^{-6})(100)^2 + \frac{1}{2}(2 \times 10^{-6})(180)^2 \qquad (U = \frac{1}{2}CV^2)$$

$= 4.74 \times 10^{-2}$ J

$= 47.4$ *m*J.

(b) Electrostatic energy stored after completing the circuit,

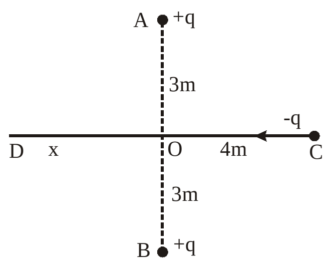
$$U_f = \frac{1}{2}(90 \times 10^{-6})^2 . \frac{1}{3 \times 10^{-6}} + \frac{1}{2}(90 \times 10^{-6})^2 . \frac{1}{2 \times 10^{-6}} + \frac{1}{2}(150 + 10^{-6})^2 . \frac{1}{2 \times 10^{-6}}$$

$$\left(U = \frac{1}{2}\frac{q}{c^2}\right)$$

$= 90 \times 10^{-4}$ J

$= 9$ mJ.

20. Two fixed positive charges, each of magnitude 5×10^{-5} C are located at points A and B, separated by a distance of 6 m. An equal and opposite charge moves towards them along the line COD, the perpendicular bisector of lien AB. The moving charge, when reaches the point C at a distance of 4 m from O, has a kinetic energy of 4 joules. Calculate the distance of the farthest point D which the negative charge will reach before returning towards C.



Solution: The kinetic energy is lost and converted to electrostatic potential energy of the system as the negative charge goes from C to D and comes to rest at D instantaneously.

Loss of KE = gain in PE

$$4 = U_f - U_i$$

$$4 = \left[\frac{qq}{4\pi\varepsilon_0(6)} + \frac{2q(-q)}{4\pi\varepsilon_0\sqrt{9 + x^2}}\right] - \left[\frac{qq}{4\pi\varepsilon_0(6)} + \frac{2q(-q)}{4\pi\varepsilon_0\sqrt{9 + 16}}\right]$$

$$4 = \frac{2q^2}{4\pi\varepsilon_0}\left[\frac{1}{5} - \frac{1}{\sqrt{9 + x^2}}\right]$$

$$4 = 2(5\times 10^{-5})^2(9\times 10^9)\left(\frac{1}{5} - \frac{1}{\sqrt{9 + x^2}}\right)$$

$$4 = 9 - \frac{45}{\sqrt{9 + x^2}}$$

$$x = \sqrt{72} = 8.48\ \text{m}.$$