Lesson-5
DETERMINANTS

Determinants

A determinant of order two is written as $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ and is equal to $a_{11}a_{22} - a_{12}a_{21}$

A determinant of order three is written as $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

where $R_1 = (a_{11}, a_{12}, a_{13})$, $R_2 = (a_{21}, a_{22}, a_{23})$, $R_3 = (a_{31}, a_{32}, a_{33})$ are its rows, and

$C_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$, $C_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}$, $C_3 = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$ are its columns.

$$\Delta = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

Minor and Cofactors

The determinant obtained by deleting the ith row and jth column is denoted by M_{ij} and is called the minor of element a_{ij}. The co-factor of the element a_{ij} is denoted by C_{ij} and is given by $(-1)^{i+j} M_{ij}$. If we apply the appropriate sign to the minor of an element, we get its cofactor ; these signs (for 3rd order determinant) are as $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

Properties of Determinants

Determinants have some properties that are useful as they permit to generate equal determinants with different and simpler configurations of entries (elements).

- ❖ **Reflection Property :** The determinant remains unaltered if its rows are changed into columns and the columns into rows.

- ❖ $\begin{vmatrix} 0 & 0 & 0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} 0 & \alpha_1 & \alpha_2 \\ 0 & \beta_1 & \beta_2 \\ 0 & \gamma_1 & \gamma_2 \end{vmatrix} = 0.$ Here all elements in a row or column are zero.

$$\begin{vmatrix} a & 0 & 0 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}$$

. If all elements except one element of a row are reduced to zero then we get determinant of an order less by one. Similar result is valid for a column.

- ❖ If the elements of a row (column) are proportional or identical to the elements of some other row (column), then the determinant is zero.

Example: $\begin{vmatrix} a_1 & a_2 & a_3 \\ ka_1 & ka_2 & ka_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0$

- ❖ The interchange of any two adjacent rows (columns) of the determinant changes its sign.
- ❖ If all the elements of a row (column) of a determinant are multiplied by a non-zero constant, then the determinant gets multiplied by the same constant.
- ❖ **Property of Invariance**

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+\alpha b_1+\beta c_1 & b_1 & c_1 \\ a_2+\alpha b_2+\beta c_2 & b_2 & c_2 \\ a_3+\alpha b_3+\beta c_3 & b_3 & c_3 \end{vmatrix}$$

i.e. a determinant remains unaltered under an operation of the form

$C_i \to C_i + \alpha C_j + \beta C_k$ where $j, k \neq i$ or an operation of the form

$R_i \to R_i + \alpha R_j + \beta R_k$ $j, k \neq i$.

Sum of Determinants

If each element in a row (or column) of a determinant is written as the sum of two or more terms, then the determinant can be written as the sum of two or more determinants

Example: $\begin{vmatrix} a_1+b_1 & c_1 & d_1 \\ a_2+b_2 & c_2 & d_2 \\ a_3+b_3 & c_3 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$

Product of two determinants

Two determinants of the same order can be multiplied to get a determinant of the same order.

If $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$

Then $\Delta_1\Delta_2 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$

$$\begin{vmatrix} a_1x_1+b_1x_2+c_1x_3 & a_1y_1+b_1y_2+c_1y_3 & a_1z_1+b_1z_2+c_1z_3 \\ a_2x_1+b_2x_2+c_2x_3 & a_2y_1+b_2y_2+c_2y_3 & a_2z_1+b_2z_2+c_2z_3 \\ a_3x_1+b_3x_2+c_3x_3 & a_3y_1+b_3y_2+c_3y_3 & a_3z_1+b_3z_2+c_3z_3 \end{vmatrix}$$

Derivative of a Determinant

A determinant can be differentiated as follows :

If $\Delta(x) = \begin{vmatrix} u(x) & v(x) & w(x) \\ p(x) & q(x) & r(x) \\ \lambda(x) & \mu(x) & \beta(x) \end{vmatrix}$, then

$$\Delta'(x) = \begin{vmatrix} u'(x) & v'(x) & w'(x) \\ p(x) & q(x) & r(x) \\ \lambda(x) & \mu(x) & \beta(x) \end{vmatrix} + \begin{vmatrix} u(x) & v(x) & w(x) \\ p'(x) & q'(x) & r'(x) \\ \lambda(x) & \mu(x) & \beta(x) \end{vmatrix} + \begin{vmatrix} u(x) & v(x) & w(x) \\ p(x) & q(x) & r(x) \\ \lambda'(x) & \mu'(x) & \beta'(x) \end{vmatrix}$$

Differentiation can also be done columnwise.

Solution of a System of Linear Equation.

- **Non – Homogeneous System**

Cramer's Rule: Consider a system of simultaneous linear equations in three variables x, y, z ;

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \qquad ...(1)$$

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

$$\Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \quad \Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

where Δ_x, Δ_y, Δ_z are obtained by replacing first, second and third columns of Δ respectively, by column $\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$,

We discuss four cases :

❖ If $\Delta \neq 0$ the system has a **unique solution**, which is given by **Cramer's Rule :**

$$x = \frac{\Delta_x}{\Delta}, \quad y = \frac{\Delta_y}{\Delta}, \quad z = \frac{\Delta_z}{\Delta}$$

- If $\Delta \neq 0$ and $\Delta_x = \Delta_y = \Delta_z = 0$, then the system has a **trivial solution** ; ($x = y = z = 0$)
- If $\Delta = 0$ and at least one of the determinants Δ_x, Δ_y, Δ_z is non-zero, then the system is **inconsistent** i.e. it has no solution.
- If $\Delta = 0$ and $\Delta_x = \Delta_y = \Delta_z = 0$, then the system may have **infinite number of solutions** (non-trivial).

• Homogeneous – System

If $d_1 = d_2 = d_3 = 0$, then the system of equations (1) is known as homogenous system of equations.

$$\begin{aligned} a_1x + b_1y + c_1z &= 0 \\ a_2x + b_2y + c_2z &= 0 \quad \quad \ldots(2) \\ a_3x + b_3y + c_3z &= 0 \end{aligned}$$

For this system of equations, $\Delta_x = \Delta_y = \Delta_z = 0$; therefore, the system has two types of solutions :

- **Trivial Solution** : If $\Delta \neq 0$, then the only solution is $x = y = z = 0$.
- **Non-Trivial Solution** : The system (2) has a non-trivial solution (i.e. at least one of x, y, z is different from zero) only if $\Delta = 0$.

Some Results

1. $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc)$

   If $a + b + c = 0$ then $\Delta = 0$.

2. $\Delta = \begin{vmatrix} a & p & q \\ 0 & b & r \\ 0 & 0 & c \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ p & b & 0 \\ q & r & c \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$

3. $\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & bc & b+c \\ 1 & ca & c+a \\ 1 & cb & a+b \end{vmatrix} = (a-b)(b-c)(c-a)$

4. $\begin{vmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{vmatrix} = 0$

5. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$

6. $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a - b)(b - c)(c - a)(a + b + c)$$

7. $$\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a - b)(b - c)(c - a)(ab + bc + ca)$$

SOLVED EXAMPLES

Ex.1: Prove that

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = 1 + a^2 + b^2 + c^2.$$

Sol.: Multiply columns 1st, 2nd and 3rd by a, b, c respectively and divide the whole determinant by product of a, b and c:

$$\Delta = \frac{1}{abc}\begin{vmatrix} a^3+a & ab^2 & ac^2 \\ a^2b & b^3+b & bc^2 \\ a^2c & b^2c & c^3+c \end{vmatrix}$$

$$= \frac{abc}{abc}\begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix}$$

[by taking a, b, c common from first, second and third rows respectively]

$$= \begin{vmatrix} a^2+b^2+c^2+1 & b^2 & c^2 \\ a^2+b^2+c^2+1 & b^2+1 & c^2 \\ a^2+b^2+c^2+1 & b^2 & c^2+1 \end{vmatrix}; \ [C_1 \to C_1 + C_2 + C_3]$$

$$= (a^2 + b^2 + c^2 + 1)\begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix}$$

$$= (a^2 + b^2 + c^2 + 1)\begin{vmatrix} 1 & b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}; \ (R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1)$$

$$= (a^2 + b^2 + c^2 + 1)[1(1 - 0)]$$

$$= 1 + a^2 + b^2 + c^2$$

Ex.2: Show that $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2\,(a + b + c)^3$

Sol.: $\Delta = \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix}$ by $C_1 \to C_1 + C_2 + C_3$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & 0 & c+a+b \end{vmatrix}$$ [by $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

$= 2(a + b + c)[1\{(b + c + a)^2 - 0\}]$

$= 2(a + b + c)(a + b + c)^2$

$= 2(a + b + c)^3$

Ex.3: If $\Delta = \begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix} = 0$ and a, b, c are distinct, show that either α is a root of $ax^2 + 2bx + c = 0$, or a, b, c are in G.P.

Sol.: Applying $R_3 \to R_3 - \alpha R_1 - R_2$, we get

$$\Delta = \begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ 0 & 0 & -(a\alpha^2+b\alpha+b\alpha+c) \end{vmatrix}$$

$= -(a\alpha^2 + 2b\alpha + c)(ac - b^2)$; [by expanding along R_3]

Now $\Delta = 0 \Rightarrow$ either $(a\alpha^2 + 2b\alpha + c) = 0$ or $b^2 - ac = 0$

i.e. either α is root of $ax^2 + 2bx + c = 0$ or a, b, c are in G.P.

Ex.4: Solve the following system of equations by Cramer's Rule

$2x - y + 3z = 9$

$x + y + z = 6$

$x - y + z = 2.$

Sol.: Here $\Delta = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 2(1 + 1) + 1(1 - 1) + 3(-1 - 1) = -2,$

$$\Delta_x = \begin{vmatrix} 9 & -1 & 3 \\ 6 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = 9(1 + 1) + 1(6 - 2) + 3(-6 - 2) = -2$$

$$\Delta_y = \begin{vmatrix} 2 & 9 & 3 \\ 1 & 6 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 2(6 - 2) - 9(1 - 1) + 3(2 - 6) = -4$$

$$\Delta_z = \begin{vmatrix} 2 & -1 & 9 \\ 1 & 1 & 6 \\ 1 & -1 & 2 \end{vmatrix} = 2(2 + 6) + 1(2 - 6) + 9(-1 -1) = -6$$

By Cramer's Rule

$$x = \frac{\Delta_x}{\Delta} = 1,\ y = \frac{\Delta_y}{\Delta} = 2,\ z = \frac{\Delta_z}{\Delta} = 3$$

Ex.5: For what value of k the following system of equations possess non-trivial solution; also find all the solutions of the system for that value of k.

$$x + y - kz = 0$$
$$3x - y - 2z = 0$$
$$x - y + 2z = 0.$$

Sol.: For non-trivial solution, $\Delta = 0$

$$\Rightarrow \begin{vmatrix} 1 & 1 & -k \\ 3 & -1 & -2 \\ 1 & -1 & 2 \end{vmatrix} = 0$$

$$\Rightarrow 1(-2 -2) -1(6 + 2) -k(-3 + 1) = 0$$

$$-4 - 8 + 2k = 0 \quad \Rightarrow k = 6$$

Putting the value of k in the system of linear equation, we get

$$x + y - 6z = 0 \quad \text{...(i)}$$
$$3x - y - 2z = 0 \quad \text{...(ii)}$$
$$x - y + 2z = 0 \quad \text{...(iii)}$$

Adding (i) and (ii) $\Rightarrow$ $4x - 8z = 0$ $\therefore$ $z = \frac{x}{2}$

Putting the value of z in (iii), we get

$$x - y + x = 0 \qquad \therefore \quad y = 2x$$

Thus when $k = 6$, solution of given system of linear equations will be $x = t$, $y = 2t$, $z = \frac{t}{2}$ where t is an arbitrary number. Thus number of solutions of the system is infinite.

Ex. 6: Prove that the value of $\begin{vmatrix} 1 & bc & bc(b+c) \\ 1 & ca & ca(a+c) \\ 1 & ab & ab(a+b) \end{vmatrix}$ is independent of a, b, c.

Sol.: Multiply R_1, R_2, R_3 by a, b, c respectively and hence divide it by abc,

$$\therefore \quad \Delta = \frac{1}{abc}\begin{vmatrix} a & abc & abc(b+c) \\ b & abc & abc(c+a) \\ c & abc & abc(a+b) \end{vmatrix}$$

$$= \frac{(abc)(abc)}{abc}\begin{vmatrix} a & 1 & b+c \\ b & 1 & c+a \\ c & 1 & a+b \end{vmatrix}$$

$$= abc\begin{vmatrix} a & 1 & b+c \\ b & 1 & c+a \\ c & 1 & a+b \end{vmatrix} = abc\begin{vmatrix} a+b+c & 1 & b+c \\ a+b+c & 1 & c+a \\ a+b+c & 1 & a+b \end{vmatrix} \quad \text{(by using } C_1 \to C_1 + C_3\text{)}$$

$$= abc(a+b+c)\begin{vmatrix} 1 & 1 & b+c \\ 1 & 1 & c+a \\ 1 & 1 & a+b \end{vmatrix}$$

$= 0.$

Hence value of the determinant is independent of a, b, c.

Ex.7: For a fixed positive integer n, if $\Delta = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$ then show that $\left[\frac{\Delta}{(n!)^3} - 4\right]$ is divisible by n.

Sol.: $$\Delta = (n!)^3\begin{vmatrix} 1 & n+1 & (n+2)(n+1) \\ n+1 & (n+2)(n+1) & (n+3)(n+2)(n+1) \\ (n+2)(n+1) & (n+3)(n+2)(n+1) & (n+4)(n+3)(n+2)(n+1) \end{vmatrix}$$

Taking $(n + 1)$ and $(n + 1)(n + 2)$ common from C_2 and C_3 respectively, we get

$$\Delta = (n!)^3(n+2)(n+1)^2\begin{vmatrix} 1 & 1 & 1 \\ n+1 & n+2 & n+3 \\ (n+2)(n+1) & (n+3)(n+2) & (n+4)(n+3) \end{vmatrix}$$

Apply $C_3 \to C_3 - C_1$ and $C_2 \to C_2 - C_1$; then,

$$\Delta = (n!)^3(n+1)^2(n+2)\begin{vmatrix} 1 & 0 & 0 \\ n+1 & 1 & 2 \\ (n+2)(n+1) & 2(n+2) & 4n+10 \end{vmatrix}$$

$= (n!)^3(n + 1)^2(n + 2)[4n + 10 - 4(n + 2)]$

$= (n!)^3(n + 1)^2(n + 2).2$

$\Delta = (n!)^3(n^2 + 2n + 1)(2n + 4)$

$= (n!)^3(2n^3 + 8n^2 + 10n + 4)$

$\therefore \left[\frac{\Delta}{(n!)^3} - 4\right] = 2n^3 + 8n^2 + 10n$

$2n(n^2 + 4n + 5)$, which is divisible by n.

Ex.8: Show that $\begin{vmatrix} {}^xC_r & {}^xC_{r+1} & {}^xC_{r+2} \\ {}^yC_r & {}^yC_{r+1} & {}^yC_{r+2} \\ {}^zC_r & {}^zC_{r+1} & {}^zC_{r+2} \end{vmatrix} = \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+2} \end{vmatrix}$

Sol.: As ${}^xC_r + {}^xC_{r+1} = {}^{x+1}C_{r+1}$

and ${}^{x+1}C_{r+1} + {}^{x+1}C_{r+2} = {}^{x+2}C_{r+2}$

L.H.S. $= \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+1}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+1}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+1}C_{r+2} \end{vmatrix}$ (Apply $C_3 \to C_3 + C_2$ and then $C_2 \to C_2 + C_1$)

On applying $C_3 \to C_3 + C_2$ we get

L.H.S. $= \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+2} \end{vmatrix}$ = R.H.S.

Ex.9: Show that $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = \begin{vmatrix} a^2 & c^2 & 2ac-b^2 \\ 2ab-c^2 & b^2 & a^2 \\ b^2 & 2bc-a^2 & c^2 \end{vmatrix}$

Sol.: Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Replacing each element of Δ by its cofactor, we get determinant of cofactors of Δ as

$$\Delta_1 = \begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} \quad ...(i)$$

$$\Delta^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \begin{vmatrix} a & -c & b \\ b & -a & c \\ c & -b & a \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \begin{vmatrix} a & b & c \\ -c & -a & -b \\ b & c & a \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & c^2 & 2ac-b^2 \\ 2ab-c^2 & b^2 & a^2 \\ b^2 & 2bc-a^2 & c^2 \end{vmatrix} \quad ...(ii)$$

From (i) and (ii), we get the required result.

Ex.10: For all values of *A*, *B*, *C* and *P*, *Q*, *R*, show that

$$\Delta = \begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix} = 0$$

Sol.: The given determinant can be written as product of two determinants as follows :

$$\Delta = \begin{vmatrix} \cos A\cos P+\sin A\sin P & \cos A\cos Q+\sin A\sin Q & \cos A\cos R+\sin A\sin R \\ \cos B\cos P+\sin B\sin P & \cos B\cos Q+\sin B\sin Q & \cos B\cos R+\sin B\sin R \\ \cos C\cos P+\sin C\sin P & \cos C\cos Q+\sin C\sin Q & \cos C\cos R+\sin C\sin R \end{vmatrix}$$

$$= \begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix} \begin{vmatrix} \cos P & \cos Q & \cos R \\ \sin P & \sin Q & \sin R \\ 0 & 0 & 0 \end{vmatrix}$$

= (0) (0) = 0.

Ex.11: If $f(x) = \begin{vmatrix} (1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3} \\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3} \\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3} \end{vmatrix}$ (where a_i's and b_j's $\in N$), find the coefficient of x in the expansion of $f(x)$.

Sol.: Let $f(x) = c_0 + c_1x + c_2x^2 + \ldots.$; then,

$f'(x) = c_1 + 2c_2x + 3c_3x^2 + \ldots.$

$$\text{Also } f'(x) = \begin{vmatrix} a_1b_1(1+x)^{a_1b_1-1} & a_1b_2(1+x)^{a_1b_2-1} & a_1b_3(1+x)^{a_1b_3-1} \\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3} \\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3} \end{vmatrix}$$

$$+ \begin{vmatrix} (1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3} \\ a_2b_1(1+x)^{a_2b_1-1} & a_2b_2(1+x)^{a_2b_2-1} & a_2b_3(1+x)^{a_2b_3-1} \\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3} \end{vmatrix}$$

$$+ \begin{vmatrix} (1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3} \\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3} \\ a_3b_1(1+x)^{a_3b_1-1} & a_3b_2(1+x)^{a_3b_2-1} & a_3b_3(1+x)^{a_3b_3-1} \end{vmatrix}$$

$$\therefore \quad c_1 = f'(0) = \begin{vmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{vmatrix}$$

= 0 + 0 + 0

= 0

Ex.12: Prove that $\Delta = \begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix} = (bc+ca+ab)^3$.

Sol.: Multiply R_1, R_2, R_3 by a, b, c respectively and divide the determinant by abc; then,

$$\Delta = \frac{1}{abc}\begin{vmatrix} -abc & ab^2+abc & ac^2+abc \\ a^2b+abc & -abc & bc^2+abc \\ a^2c+abc & b^2c+abc & -abc \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we get

$$\Delta = \frac{1}{abc}\begin{vmatrix} a[ab+ac+bc] & b[ab+bc+ac] & c[ac+bc+ab] \\ a^2b+abc & -abc & bc^2+abc \\ a^2c+abc & b^2c+abc & -abc \end{vmatrix}$$

$$\Delta = abc\frac{(ab+bc+ca)}{abc}\begin{vmatrix} 1 & 1 & 1 \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & -ab \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$\Delta = (ab+bc+ca)\begin{vmatrix} 1 & 0 & 0 \\ ab+bc & -(ac+bc+ab) & 0 \\ ac+bc & 0 & -(ab+ac+bc) \end{vmatrix}$$

$= (ab + bc + ca)\,(ab + bc + ac)^2$

$= (ab + bc + ca)^3$.

Ex.13: If (a_r, b_r), $r = 1, 2, 3$, be the vertices of a triangle, prove that

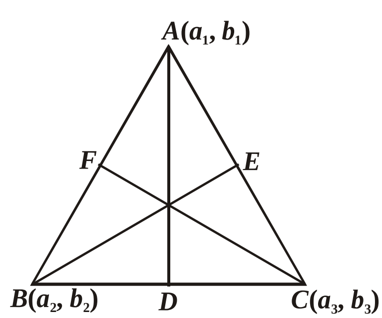
$$\Delta = \begin{vmatrix} a_2-a_3 & b_2-b_3 & a_1(a_2-a_3)+b_1(b_2-b_3) \\ a_3-a_1 & b_3-b_1 & a_2(a_3-a_1)+b_2(b_3-b_1) \\ a_1-a_2 & b_1-b_2 & a_3(a_1-a_2)+b_3(b_1-b_2) \end{vmatrix} = 0 \qquad \text{...(i)}$$

and hence show that the altitudes of a triangle are concurrent .

Sol.: Applying $R_1 \to R_1 + R_2 + R_3$

$$\Delta = \begin{vmatrix} 0 & 0 & 0 \\ a_3-a_1 & b_3-b_1 & a_2(a_3-a_1)+b_2(b_3-b_1) \\ a_1-a_2 & b_1-b_2 & a_3(a_1-a_2)+b_3(b_1-b_2) \end{vmatrix}$$

$= 0$



$\therefore$ Eq. of altitude AD is : $y - b_1 = -\dfrac{a_2 - a_3}{b_2 - b_3}(x - a_1)$

or $x(a_2 - a_3) + y(b_2 - b_3) = a_1(a_2 - a_3) + b_1(b_2 - b_3)$ ...(ii)

Similarly eqs of altitudes BE and CF are

$x(a_3 - a_1) + y(b_3 - b_1) = a_2(a_3 - a_1) + b_2(b_3 - b_1)$ ...(iii)

$x(a_1 - a_2) + y(b_1 - b_2) = a_3(a_1 - a_2) + b_3(b_1 - b_2)$ ...(iv)

Altitudes (ii), (iii), (iv) are concurrent, since the determinant given by L.H.S of (i) is zero.

Ex.14: Find values of c for which the equations

$2x + 3y = 3$

$(c + 2)x + (c + 4)y = c + 6$

$(c + 2)^2x + (c + 4)^2y = (c + 6)^2$

are consistent and hence solve the equation.

Sol.: The equation will be consistent, if

$$\begin{vmatrix} 2 & 3 & 3 \\ c+2 & c+4 & c+6 \\ (c+2)^2 & (c+4)^2 & (c+6)^2 \end{vmatrix} = 0$$

Applying $C_3 \rightarrow C_3 - C_2$, we get

$$\begin{vmatrix} 2 & 3 & 0 \\ c+2 & c+4 & 2 \\ (c+2)^2 & (c+4)^2 & 4(c+5) \end{vmatrix} = 0$$

Solving, we get $c^2 + 10c = 0$

or $c = 0, -10$ ...(i)

If $c = 0$, the system of equations becomes

$$\begin{cases} 2x + 3y = 3 \\ 2x + 4y = 6 \end{cases} \Rightarrow x = -3, y = 3 \quad ...(ii)$$

$x + 4y = 9$

If $c = -10$, the system of equations becomes

$$\begin{cases} 2x + 3y = 3 \\ -8x - 6y = -4 \end{cases} \Rightarrow x = -1/2, \; y = 4/3 \quad ...(iii)$$

$16x + 9y = 4$

Hence solutions are given by (ii) and (iii).

Ex.15: Let λ and α be real. Find the set of all values of λ and α for which the system of linear equations

$$\lambda x + (\sin \alpha)y + (\cos \alpha)z = 0$$

$$x + (\cos \alpha)y + (\sin \alpha)z = 0$$

$$-x + (\sin \alpha)y - (\cos \alpha)z = 0$$

has a non – trivial solution. If $\lambda = 1$, find all values of α.

Sol.: For non – trivial solution, condition is $\Delta = 0$.

$$\Delta = \begin{vmatrix} \lambda & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

or $\lambda[-\cos^2\alpha - \sin^2\alpha] - \sin \alpha[-\cos \alpha + \sin \alpha] + \cos \alpha[\sin \alpha + \cos \alpha] = 0$

or $\lambda = \sin 2\alpha + \cos 2\alpha$

$\therefore \quad \alpha \in R \; ; \; |\lambda| \le \sqrt{2}$

If $\lambda = 1$, then $1 = \sin 2\alpha + \cos 2\alpha$

or $\cos\left(2\alpha - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4}$

$\Rightarrow \quad 2\alpha - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4} \; : n \in I$

$\Rightarrow \quad \alpha = n\pi \pm \frac{\pi}{8} + \frac{\pi}{8} \; : n \in I$

BASIC LEVEL ASSIGNMENT

Evaluate :

1. If $\begin{vmatrix} 6i & 3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$ then find x and y

2. $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$

3. $\begin{vmatrix} x-y-z & 2x & 2x \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix}$

4. Find the number of distinct real roots of the equation

$\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$ in the $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

5. $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

Prove the following results :

6. $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = x^2(x + a + b + c)$

7. $\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix} = 2(a+b)(b+c)(c+a)$

8. Solve the system of equations :

$3x - 4y + 5z = -6$

$x + y - 2z = -1$

$2x + 3y + z = 5$

9. Show that the following equations have infinite number of solutions and find them :

$2x - 3y - z = 0$

$x + 3y - 2z = 0$

$x - 3y = 0.$

10. If $\begin{vmatrix} \lambda^2+\lambda^3 & \lambda-1 & \lambda+3 \\ \lambda+1 & 1-2\lambda & \lambda-4 \\ \lambda-2 & \lambda+4 & 3\lambda \end{vmatrix} = k\lambda^5 + p\lambda^4 + q\lambda^3 + r\lambda^2 + s\lambda + t$ be an identity in λ, where p, q, r, s, k and t are constants, find the value of t.

11. Find the value of k such that following system of equations possess a non-trivial solution (*i.e.* not all zero solution). Also find the nontrivial solutions.

$x + ky + 3z = 0$

$3x + ky - 2z = 0$

$2x + 3y - 4z = 0$

12. If $x = cy + bz$, $y = az + cx$, $z = bx + ay$ where x, y, z are not all zero, prove that

$a^2 + b^2 + c^2 + 2abc = 1.$

13. Let a, b, c be positive and not all equal. Show that the value of the determinant $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is negative.

14. If $\Delta = \begin{vmatrix} x & 1 & x^2 \\ x+2 & 2x+3 & x \\ x^2 & x^3+1 & 2x^4+1 \end{vmatrix}$ find the value of $\dfrac{d\Delta}{dx}$.

15. Let the three digit numbers $A28$, $3B9$ and $62C$ where A, B and C are integers between 0 and 9, be divisible by a fixed integer k. Show that the determinant $\begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$ is also divisible by k.

16. For what values of a and b, the system of equation $2x + ay + 6z = 8$, $x + 2y + bz = 5$, $x + y + 3z = 4$ has

(i) no solution (ii) a unique solution (iii) infinitely many solutions.

ADVANCED LEVEL ASSIGNMENT

1. Let a, b, c be real numbers with $a^2 + b^2 + c^2 = 1$. Then show that

$$\begin{vmatrix} ax-by-c & bx+ay & cx+a \\ bx+ay & -ax+by-c & cy+b \\ cx+a & cy+b & -ax-by+c \end{vmatrix} = 0$$ represents a straight line

2. Show that $\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$

3. Show that $\begin{vmatrix} 1-x & a & a^2 \\ a & a^2-x & a^3 \\ a^2 & a^3 & a^4-x \end{vmatrix} = x^2(1+a^2+a^4) - x^3$

4. Without expanding the determinant at any stage, show that

$$\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = xA + B$$

where A , B are determinants of the third order not containing x.

5. Evaluate $\Delta = \begin{vmatrix} {}^xC_1 & {}^xC_2 & {}^xC_3 \\ {}^yC_1 & {}^yC_2 & {}^yC_3 \\ {}^zC_1 & {}^zC_2 & {}^zC_3 \end{vmatrix}$

6. If α, β be the roots of the equation $ax^2 + bx + c = 0$ and $s_n = \alpha^n + \beta^n : n \geq 1$ evaluate

$$\begin{vmatrix} 3 & 1+s_1 & 1+s_2 \\ 1+s_1 & 1+s_2 & 1+s_3 \\ 1+s_2 & 1+s_3 & 1+s_4 \end{vmatrix}$$

7. If lines $px + by = c$, $ax + qy = c$ and $ax + by = r$ $(a \neq p,\ b \neq q,\ c \neq r)$ are concurrent then find the value of $\dfrac{p}{p-a} + \dfrac{q}{q-b} + \dfrac{r}{r-c}$.

8. If $x_r \neq 0$; $r = 1, 2, 3$ then prove that

$$\begin{vmatrix} x_1 + a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & x_2 + a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & x_3 + a_3b_3 \end{vmatrix} = x_1x_2x_3 \left(1 + \frac{a_1b_1}{x_1} + \frac{a_2b_2}{x_2} + \frac{a_3b_3}{x_3}\right)$$

9. Let $f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$. Prove that $\int_0^{\pi/2} f(x)\, dx = -\left(\frac{\pi}{4} + \frac{8}{15}\right)$

10. Let $a > 0$, $d > 0$. Find the value of $\begin{vmatrix} \frac{1}{a} & \frac{1}{a(a+d)} & \frac{1}{(a+d)(a+2d)} \\ \frac{1}{a+d} & \frac{1}{(a+d)(a+2d)} & \frac{1}{(a+2d)} \frac{1}{(a+3d)} \\ \frac{1}{a+2d} & \frac{1}{(a+2d)(a+3d)} & \frac{1}{(a+3d)(a+4d)} \end{vmatrix}$

11. Prove that $\begin{vmatrix} 2 & \alpha + \beta + \gamma + \delta & \alpha\beta + \gamma\delta \\ \alpha + \beta + \gamma + \delta & 2(\alpha + \beta)(\gamma + \delta) & \alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) \\ \alpha\beta + \gamma\delta & \alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) & 2\alpha\beta\gamma\delta \end{vmatrix} = 0$

12. If $(x_1 - x_2)^2 + (y_1 - y_2)^2 = a^2$, $(x_2 - x_3)^2 + (y_2 - y_3)^2 = b^2$, $(x_1 - x_3)^2 + (y_1 - y_3)^2 = c^2$ then prove that

$$4 \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = (a + b + c)(a + b - c)(a - b + c)(-a + b + c).$$

13. Prove that $\begin{vmatrix} \operatorname{cosec} x & 1 & 0 \\ 1 & 2 \operatorname{cosec} x & 1 \\ 0 & 1 & 2 \operatorname{cosec} x \end{vmatrix} \geq 1$ for all $x \in (0, \pi)$.

14. Let $\Delta = \begin{vmatrix} \sin x & \sin(x+h) & \sin(x+2h) \\ \sin(x+2h) & \sin x & \sin(x+h) \\ \sin(x+h) & \sin(x+2h) & \sin x \end{vmatrix}$. Evaluate $\lim_{h \to 0} \left(\frac{\Delta}{h^2}\right)$.

15. If the following system of linear equations in x, y

$ax + hy + g = 0$

$hx + by + f = 0$

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = t$, admits of a solution, then prove that

$$t = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \div \begin{vmatrix} a & h \\ h & b \end{vmatrix}.$$

16. Prove that for all values of θ

$$\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta+\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) & \sin\left(2\theta+\frac{4\pi}{3}\right) \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta-\frac{4\pi}{3}\right) \end{vmatrix} = 0.$$

17. Let n and r be two positive integers and that $n \geq r + 2$. Suppose

$$\Delta(n, r) = \begin{vmatrix} {}^{n}C_{r} & {}^{n}C_{r+1} & {}^{n}C_{r+2} \\ {}^{n+1}C_{r} & {}^{n+1}C_{r+1} & {}^{n+1}C_{r+2} \\ {}^{n+2}C_{r} & {}^{n+2}C_{r+1} & {}^{n+2}C_{r+2} \end{vmatrix}$$

Show that $\Delta(n, r) = \dfrac{{}^{n+2}C_3}{{}^{r+2}C_3}\,\Delta(n-1, r-1)$.

Hence or otherwise, prove that

$$\Delta(n, r) = \frac{({}^{n+2}C_3)\,({}^{n+1}C_3)\ldots.({}^{n-r+3}C_3)}{({}^{r+2}C_3)\,({}^{r+1}C_3)\ldots..({}^{3}C_3)}.$$

18. Let a, b, c be real numbers with $a^2 + b^2 + c^2 = 1$. Show that the equation

$$\begin{vmatrix} ax-by-c & bx+ay & cx+a \\ bx+ay & -ax+by-c & cy+b \\ cx+a & cy+b & -ax-by+c \end{vmatrix} = 0$$ represents a straight line.

19. Suppose $f(x)$ is a function satisfying the following conditions

(a) $f(0) = 2, f(1) = 1,$

(b) f ha a minimum value at $x = 5/2$

(c) for all x, $f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2ab+1 & 2ax+b \end{vmatrix}$

where a, b are some constants. Determine the constants $a, b,$ and the function $f(x)$

20. If $f(x) = \begin{vmatrix} x^n & \sin x & \cos x \\ n! & \sin\left(\frac{n\pi}{2}\right) & \cos\left(\frac{n\pi}{2}\right) \\ a & a^2 & a^3 \end{vmatrix}$ then find the value of $\dfrac{d^n}{dx^n}[f(x)]$ at $x = 0$.

OBJECTIVE ASSIGNMENT

Choose the correct option(s) in the following :

1. If A, B, C are angles of a triangle, then the value of

$$\begin{vmatrix} e^{2iA} & e^{-iC} & e^{-iB} \\ e^{-iC} & e^{2iB} & e^{-iA} \\ e^{-iB} & e^{-iA} & e^{2iC} \end{vmatrix} \text{ is}$$

(a) 1 (b) –1 (c) –2 (d) –4

2. Given $A = \begin{vmatrix} a & b & 2c \\ d & e & 2f \\ l & m & 2n \end{vmatrix}$ and $B = \begin{vmatrix} f & 2d & e \\ 2n & 4l & 2m \\ c & 2a & b \end{vmatrix}$ then

(a) $2A + B = 0$ (b) $2A - B = 0$ (c) $A + 2B = 0$ (d) $A - 2B = 0$

3. The value of determinant $\begin{vmatrix} x+1 & x+2 & x+4 \\ x+3 & x+5 & x+8 \\ x+7 & x+10 & x+14 \end{vmatrix}$ is

(a) –2 (b) $x^2 + 2$ (c) 2 (d) none of these

4. If 7 and 2 are roots of the equation $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$, then the third root is

(a) –9 (b) 14 (c) ½ (d) none of these

5. If $f(x, y) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$, then $f(x, y)$ is

(a) not divisible by x (b) not divisible by y

(c) divisible by both x and y (d) divisible by neither x nor y

6. If the equations $x = ay + z$, $y = az + x$, $z = ax + y$ are consistent having non-trivial solution then

(a) $a^3 = 1$ (b) $a^3 + 1 = 0$ (c) $a = 2$ (d) none of these

7. Given the system of equations $kx + y + z = 1$, $x + ky + z = k$, $x + y + kz = k^2$, for what value(s) of k does this system have no solution

(a) 0 (b) 1 (c) –1 (d) –2

8. The system of equations $x - 2y + z = 0$, $kx - y + 2z = 0$, $2x - y + z = 0$ has a non trivial solution if k and $\left(\frac{x}{z}, \frac{y}{z}\right)$ are equal to

(a) $5, \left(-\frac{1}{3}, \frac{1}{3}\right)$ (b) $6, \left(-\frac{1}{6}, \frac{1}{6}\right)$ (c) $5, \left(\frac{1}{3}, \frac{-1}{5}\right)$ (d) none of these

9. The system of equation $ax + y = 2$, $x + y = 2a$ possesses infinitely many solutions when a is

(a) –1 (b) 1 (c) 8 (d) none of these

10. The system of equations $ax + 4y + z = 0$, $bx + 3y + z = 0$, $cx + 2y + z = 0$ have non trivial solutions if a, b, c are in

(a) A.P. (b) G.P. (c) H.P. (d) none of these

11. If $f(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1) \end{vmatrix}$ then $f(100)$ is equal to

(a) 0 (b) 1 (c) 100 (d) none of these.

12. If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$, then

(a) $x = 3, y = 1$ (b) $x = 1, y = 3$ (c) $x = 0, y = 3$ (d) none of these.

13. If $f_r(x)$, $g_r(x)$, $h_r(x)$, $r = 1, 2, 3$ are polynomials in x such that $f_r(a) = g_r(a) = h_r(a)$ $\forall$ $r = 1, 2, 3$

and $F(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$ then $F'(a) =$

(a) 0 (b) $g(a)$ (c) $f(a)\ g'(a)$ (d) none of these.

14. Let $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$ where p is a constant, then $\frac{d^3}{dx^3}[f(x)]$ at $x = 0$, is

(a) p (b) $p + p^2$ (c) $p + p^3$ (d) independent of p

15. If α, β and ν are the roots of the equation $x^3 + px + q = 0$, then the value of the determinant $\begin{vmatrix} \alpha & \beta & \nu \\ \beta & \nu & \alpha \\ \nu & \alpha & \beta \end{vmatrix}$ is equal to

(a) p (b) q (c) $p^2 - 2q$ (d) none of these

16. If $f(x) = \begin{vmatrix} \cos x & 1 & 0 \\ 1 & 2\cos x & 1 \\ 0 & 1 & 2\cos x \end{vmatrix}$ then $\int_0^{\frac{\pi}{2}} f(x)dx$ is equal to

(a) 1/4 (b) –1/3 (c) 1/2 (d) none of these.

17. If the value of the determinant $\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix}$ is positive, then $(a, b, c > 0)$

(a) $abc > 1$ (b) $abc > -8$ (c) $abc < -8$ (d) $abc > -2$

18. In a third order determinant, a_{ij} denotes the element in the ith row and the jth column. If

$a_{ij} = \begin{cases} 0 & , \; i = j \\ -1 & , \; i < j \\ 1 & , \; i > j \end{cases}$ then the value of the determinant is

(a) 0 (b) 1 (c) –1 (d) none of these .

19. If ω is a non real cube root of unity, then a root of the equation $\begin{vmatrix} x+1 & \omega & \omega^2 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix} = 0$, is

(a) $x = 0$ (b) $x = \omega$ (c) $x = \omega^2$ (d) none of these

20. $\begin{vmatrix} x+1 & x+2 & x+\lambda \\ x+2 & x+3 & x+\mu \\ x+3 & x+4 & x+\nu \end{vmatrix} = 0$ (where λ, μ, ν are in A.P), is

(a) an equation whose all roots are real (b) an identity in x.

(c) an equation with only one real root. (d) none of these.

21. If the determinant $\begin{vmatrix} \cos 2x & \sin^2 x & \cos 4x \\ \sin^2 x & \cos 2x & \cos^2 x \\ \cos 4x & \cos^2 x & \cos 2x \end{vmatrix}$ is expanded in powers of sin x then the constant term in the expansion is

(a) 1 (b) 2 (c) –1 (d) None of these

22. If $\Delta_1 = \begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} x & b \\ a & x \end{vmatrix}$ are the given determinants, then

(a) $\Delta_1 = 3(\Delta_2)^2$ (b) $(d/dx)\Delta_1 = 3\Delta_2$ (c) $(d/dx)\Delta_1 = 3\ (\Delta_2)^2$ (d) $\Delta_1 = 3\Delta_2^{3/2}$

23. Given : $a_i^2 + b_i^2 + c_i^2 = 1$ $(i = 1, 2, 3)$ and $a_i a_j + b_i b_j + c_i c_j = 0$ $(i \neq j\ ;\ i, j = 1, 2, 3)$; then the value of $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$, is

(a) 0 (b) 1/2 (c) ±1 (d) 2

24. If $\Delta_r = \begin{vmatrix} 2r & x & n(n+1) \\ 6r^2-1 & y & n^2(2n+3) \\ 4r^3-2nr & z & n^3(n+1) \end{vmatrix}$, then the value of $\sum_{r=1}^{n} \Delta_r$ is independent of

(a) x only (b) y only (c) x and z only (d) x, y, z, n

25. If $x > 0$ and $\neq 1$, $y > 0$ and $\neq 1$, $z > 0$ and $\neq 1$ then the value of $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$ is

(a) 0 (b) 1 (c) –1 (d) none of these

26. If $f(x)$ and $g(x)$ are functions such that $f(x + y) = f(x)\, g(y) + g(x)\, f(y)$, then $\begin{vmatrix} f(\alpha) & g(\alpha) & f(\alpha+\theta) \\ f(\beta) & g(\beta) & f(\beta+\theta) \\ f(\lambda) & g(\lambda) & f(\lambda+\theta) \end{vmatrix}$ is independent of

(a) α (b) β (c) γ (d) all of α, β, γ, θ

27. If $a \neq 0$, $\begin{vmatrix} x+1 & x & x \\ x & x+a & x \\ x & x & x+a^2 \end{vmatrix} = 0$ represebts

(a) a staight line parallel to x-axis (b) a straight lines parallel to y-axis

(c) a parabola (d) a circle

28. If $f(x)$ is of period 3 and $g(x)$ is of period 2, then the period of the function $F(x) = \begin{vmatrix} f(x) & f(x/3) \\ g(x) & g(x/2) \end{vmatrix}$ is

(a) 6 (b) 12 (c) 18 (d) 36

29. If $\begin{vmatrix} x & 5 & 6 \\ 5 & 6 & x \\ 6 & x & 5 \end{vmatrix} = \begin{vmatrix} 4 & x & 7 \\ x & 7 & 4 \\ 7 & 4 & x \end{vmatrix} = \begin{vmatrix} 1 & 10 & x \\ 10 & x & 1 \\ x & 1 & 10 \end{vmatrix} = 0$, then x is equal to

(a) 1 (b) –1 (c) –9 (d) –11

30. If [ ] denotes the greatest ingeter less than or equal to the real number under consideration, and $-1 \le x < 0$, $0 \le y < 1$, $1 \le z < 2$, then the value of the determinant $\begin{vmatrix} [x]+1 & [y] & [z] \\ [x] & [y]+1 & [z] \\ [x] & [y] & [z]+1 \end{vmatrix}$ is

(a) $[x]$ (b) $[y]$ (c) $[z]$ (d) none of these

MORE THAN ONE CORRECT CHOICE QUESTIONS

31. If $g(x) = \begin{vmatrix} a^{-x} & e^{x\log_e a} & x^2 \\ a^{-3x} & e^{3x\log_e a} & x^4 \\ a^{-5x} & e^{5x\log_e a} & 1 \end{vmatrix}$, then

(a) graphs of $g(x)$ is symmetrical about the origin (b) graphs of $g(x)$ is symmetrical about the y-axis

(c) $\left.\dfrac{d^4 g(x)}{dx^4}\right|_{x=0} = 0$ (d) $f(x) = g(x) \times \log\left(\dfrac{a-x}{a+x}\right)$ is an odd function

32. If $\Delta = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{vmatrix}$, then

(a) Δ is independent of θ (b) Δ is independent of ϕ

(c) Δ is a constant (d) $\left.\dfrac{d\Delta}{d\theta}\right]_{\theta=\pi/2} = 0$

33. If $f(\alpha, \beta) = \begin{vmatrix} \cos\alpha & -\sin\alpha & 1 \\ \sin\alpha & \cos\alpha & 1 \\ \cos(\alpha+\beta) & -\sin(\alpha+\beta) & 1 \end{vmatrix}$, then

(a) $f(300, 200) = f(400, 200)$ (b) $f(200, 400) = f(200, 600)$

(c) $f(100, 200) = f(200, 200)$ (d) none of these

34. The determinant D $= \begin{vmatrix} a^2+x & ab & ac \\ ab & b^2+x & bc \\ ac & bc & c^2+x \end{vmatrix}$ is divisible by

(a) x (b) x^2 (c) x^3 (d) none of these

35. If D(x) = $\begin{vmatrix} x^2+4x-3 & 2x+4 & 13 \\ 2x^2+5x-9 & 4x+5 & 26 \\ 8x^2-6x+1 & 16x-6 & 104 \end{vmatrix}$ = $ax^3 + bx^2 + cx + d$, then

(a) $a = 3$ (b) $b = 0$ (c) $c = 0$ (d) none of these

36. If $f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$, then $f(2x) - f(x)$ is divisible by

(a) x (b) a (c) $2a + 3x$ (d) x^2

37. The root of the equation $\begin{vmatrix} {}^xC_r & {}^{n-1}C_r & {}^{n-1}C_{r-1} \\ {}^{x+1}C_r & {}^nC_r & {}^nC_{r-1} \\ {}^{x+2}C_r & {}^{n+1}C_r & {}^{n+1}C_{r-1} \end{vmatrix} = 0$ are

(a) $x = n$ (b) $x = n + 1$ (c) $x = n - 1$ (d) $x = n - 2$

38. Let $f(n) = \begin{vmatrix} n & n+1 & n+2 \\ {}^nP_n & {}^{n+1}P_{n+1} & {}^{n+2}P_{n+2} \\ {}^nC_n & {}^{n+1}C_{n+1} & {}^{n+2}C_{n+2} \end{vmatrix}$, where the symbols have their usual meanings. Then $f(n)$ is divisible by

(a) $n^2 + n + 1$ (b) $(n + 1)!$ (c) $n!$ (d) none of these

39. The value of k Î R for which the system of equations $x + ky + 3z = 0$, $kx + 2y + 2z = 0$, $2x + 3y + 4z = 0$ admits of nontrivial solution is

(a) 2 (b) 5/2 (c) 3 (d) 5/4

40. D = $\begin{vmatrix} a & a^2 & 0 \\ 1 & 2a+b & (a+b) \\ 0 & 1 & 2a+3b \end{vmatrix}$ is divisible by

(a) $a + b$ (b) $a + 2b$ (c) $2a + 3b$ (d) a

MISCELLANEOUS ASSIGNMENT

Comprehension-1

Consider the system of linear equation in three variable x, y, z

$$a_1x + b_1y + c_1z = d_1\,; \quad a_2x + b_2y + c_2z = d_2; \quad a_3x + b_3y + c_3z = d_3$$

in matrix from we can write it as $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ *i.e.* AX = B

(*i*) If A is non singular matrix (*i.e.* $|A| \neq 0$), then $X = A^{-1}B$ gives unique solution for system.

(*ii*) If A is singular matrix (*i.e.* $|A| = 0$) then system will have no unique solution, if (adj. A) B = 0

(*iii*) If (adj A) B $\neq$ 0 but matrix is singular, the system has no solution *i.e.* it is inconsistent.

1. If the system of equation $x - ky - z = 0$, $kx - y - z = 0$ and $x + y - z = 0$ has a non zero solution then the possible values of k are

(a) –1, 2 (b) 1, 2 (c) 0, 1 (d) –1, 1

2. The system of linear equations

$$x + y + z = 2;\ 2x + y - z = 3;\ 3x + 2y + kz = 4$$

has a unique solution if

(a) $k \neq 0$ (b) $-1 < k < 1$ (c) $-2 < k < 2$ (d) $k = 0$

3. The system of equations

$$x + 2y + 3z = 4;\ 2x + 3y + 4z = 5;\ 3x + 4y + 5z = 6 \text{ has}$$

(a) many solution (b) no solution (c) unique solution (d) none of these

4. If the system of equation

$$x + 2y - 3z = 1;\ (p + 2)z = 3;\ (2p + 1)y + z = 2 \text{ is inconsistent, then the value of } p \text{ is}$$

(a) –2 (b) 1 (c) 0 (d) 2

Comprehension-2

A set of vector $\{(a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3)\}$ is said to be linearly independent if and only if

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \neq 0$$

otherwise the set is said to be linearly dependent. A similar result holds for $\{(a_1, a_2), (b_1, b_2)\}$.

5. If (a_1, a_2, a_3), (b_1, b_2, b_3) and (c_1, c_2, c_3) are linearly independent and

$x(a_1, a_2, a_3) + y(b_1, b_2, b_3) + z(c_1, c_2, c_3) = 0$, then

(a) $x = y = z$ (b) $x = y = z = 0$ (c) $x + y + z = 0$ (d) $x + y + z \neq 0$

6. If $(1, a, a^2)$, $(1, b, b^2)$ and $(1, c, c^2)$ are linearly independent, then

(a) $a + b + c \neq 0$ (b) $(b - a)(c - b) \neq 0$

(c) $(b - c)(c - a)(a - b) \neq 0$ (d) none of these

7. If a, b, c are distinct and $(a, a^2, a^3 + 1)$, $(b, b^2, b^3 + 1)$, $(c, c^2, c^3 + 1)$ are linearly dependent, then value of abc is

(a) -1 (b) -2 (c) 2 (d) 1

8. If $A = \begin{bmatrix} a & b \\ c & \frac{1+bc}{a} \end{bmatrix}$ $(a \neq 0)$ then $\lambda A^{-1} = \mu I - kA$, where I is the identity matrix of order 2, then match the following

A.	det (A)	(p)	$a + 2$
B.	λ	(q)	1
C.	μ	(r)	$a^2 + bc + 1$
D.	$k + 2$	(s)	a

9.

A.	Coefficient of x in $f(x) = \begin{vmatrix} x & (1+\sin x)^3 & \cos x \\ 1 & \log(1+x) & 2 \\ x^2 & 1+x^2 & 0 \end{vmatrix}$	(p)	10
B.	Maximum value of $\begin{vmatrix} 1 & 3\cos\theta & 1 \\ \sin\theta & 1 & 3\cos\theta \\ 1 & \sin\theta & 1 \end{vmatrix}$	(q)	0
C.	If a, b, c are in A.P. and $f(x) = \begin{vmatrix} x+a & x^2+1 & 1 \\ x+b & 2x^2-1 & 1 \\ x+c & 3x^2-2 & 1 \end{vmatrix}$, then $f'(0)$ is	(r)	-12
D.	If $\begin{vmatrix} x & 2 & x \\ 1 & x & 6 \\ x & x & x+1 \end{vmatrix} = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, then a_0 is	(s)	-2

INTEGER TYPE QUESTIONS

10. Maximum value of the expression $\begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1+\cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix}$ is equal to.

11. If $\begin{vmatrix} 1 & x & x^2 \\ x & x^2 & 1 \\ x^2 & 1 & x \end{vmatrix} = 3$, find the value of $\begin{vmatrix} x^3-1 & 0 & x-x^4 \\ 0 & x-x^4 & x^3-1 \\ x-x^4 & x^3-1 & 0 \end{vmatrix}$.

12. If $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$ = $(a + b + c)^m$, then m is.

13. If the equation $2x + 2y + 1 = 0$, $6x + 2y - 3 = 0$ and $ax + 2y - b = 0$ are consistent, then $a - b$ is

14. Maximum value of a second order determinant whose each entry is either zero or one is equal to.

15. If $a_1, a_2, a_3, 5, 4, a_6, a_7, a_8, a_9$ are in H.P., and D = $\begin{vmatrix} a_1 & a_2 & a_3 \\ 5 & 4 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$, then the value of [D] is (where [.] represents the greatest integer function).

16. Absolute value of sum of roots of the equation $\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix} = 0$ is

17. The value of |a| for which the system of equation

$\alpha x + y + z = \alpha - 1$; $x + \alpha y + z = \alpha - 1$; $x + y + \alpha z = \alpha - 1$; has no solution is

18. The sum of values of p for which the equations $x + y + z = 1$, $x + 2y + 4z = p$, and $x + 4y + 10z = p^2$ have a solution is

19. If D = $\begin{vmatrix} x & x+y & x+y+z \\ 2x & 3x+2y & 4x+3y+2z \\ 3x & 6x+3y & 10x+6y+3z \end{vmatrix} = 64$, then the real value of x is.

PREVIOUS YEAR QUESTIONS

IIT-JEE/JEE-ADVANCE QUESTIONS

1. The determinant $\begin{vmatrix} xp+y & x & y \\ py+z & y & z \\ 0 & xp+y & yp+z \end{vmatrix} = 0$ if

(a) x, y, z are in A.P. (b) x, y, z are in G.P.

(c) x, y, z are in H.P. (d) xy, yz, zx are in A.P.

2. The parameter on which the value of the determinant $\begin{vmatrix} 1 & a & a^2 \\ \cos(p-d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x & \sin px & \sin(p+d)x \end{vmatrix}$ does not depend upon

(a) a (b) p (c) r (d) x

3. If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$, then

(a) $x = 3, y = 1$ (b) $x = 1, y = 3$ (c) $x = 0, y = 3$ (d) $x = 0, y = 0$

4. If $f(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1) \end{vmatrix}$, then $f(100)$ is equal to

(a) 0 (b) 1 (c) 100 (d) –100

5. For all values of θ : $\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta + \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) & \sin\left(2\theta + \frac{4\pi}{3}\right) \\ \sin\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta - \frac{2\pi}{3}\right) & \sin\left(2\theta - \frac{4\pi}{3}\right) \end{vmatrix}$ is equal to

(a) 1 (b) 0 (c) –1 (d) 2

6. The number of distinct real roots of $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$ in the interval $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$ is

(a) 0 (b) 2 (c) 1 (d) 3

7. If the system of equations $x - Ky - z = 0$, $Kx - y - z = 0$, $x + y - z = 0$ has a non-zero solution, then the possible values of K are:

(a) –1, 2 (b) 1, 2 (c) 0, 1 (d) –1, 1

8. The number of values of K for which the system of equations $(K + 1)x + 8y = 4K$ and $Kx + (K + 3)y = 3K - 1$ has infinitely many solutions is

(a) 0 (b) 1 (c) 2 (d) infinite

9. Consider the system of equations

$x - 2y + 3z = -1$

$-x + y - 2z = k$

$x - 3y + 4z = 1.$

STATEMENT-1: The system of equations has no solution for $k \neq 3$.

STATEMENT-2: The determinant $\begin{vmatrix} 1 & 3 & -1 \\ -1 & -2 & k \\ 1 & 4 & 1 \end{vmatrix} \neq 0$, for $k \neq 3$.

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1

(b) Statement-1 and Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1

(c) Statement-1 is True, Statement-2 is False

(d) Statement-1 is False, Statement-2 is True

10. Let (x, y, z) be points with integer coordinates satisfying the system of homogeneous equations :

$3x - y - z = 0$

$-3x + z = 0$

$-3x + 2y + z = 0$

Then the number of such points for which $x^2 + y^2 + z^2 \leq 100$ is

$A = \begin{bmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2k-1 & \sqrt{k} \\ 1-2k & 0 & 2\sqrt{k} \\ -\sqrt{k} & -2\sqrt{k} & 0 \end{bmatrix}$

If det (adj A) + det (adj B) = 10^6, then $[k]$ is equal to

[Note: adj M denotes the adjoint of a square matrix M and $[k]$ denotes the largest integer less than or equal to k]

12. If the adjoint of a 3×3 matrix P is $\begin{bmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{bmatrix}$, then the possible value(s) of the determinant of P is (are)

(a) –2 (b) –1 (c) 1 (d) 2

DCE QUESTIONS

1. The determinant $\Delta = \begin{vmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) & \cos 2\beta \\ \sin\alpha & \cos\alpha & \sin\beta \\ -\cos\alpha & \sin\alpha & \cos\beta \end{vmatrix}$ is independent

(a) α (b) β (c) α and β (d) neither α nor β

2. The determinant $\Delta = \begin{vmatrix} a & b & a\alpha + b \\ b & c & b\alpha + c \\ a\alpha + b & b\alpha + c & 0 \end{vmatrix}$ is equal to zero if

(a) *a, b, c* are in A.P. (b) *a, b, c* are in G.P. (c) *a, b, c* are in H.P. (d) none of these

3. If $a \neq b \neq c$, one value of x which satisfies the equation $\begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix} = 0$ is given by

(a) $x = a$ (b) $x = b$ (c) $x = c$ (d) $x = 0$

4. The value of the determinant $\begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix}$ is

(a) $(a + b + c)(a^2 + b^2 + c^2)$ (b) $a^3 + b^3 + c^3 - 3abc$

(c) $(a - b)(b - c)(c - a)$ (d) none of these

5. If $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = k \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ then $k =$

(a) 1 (b) 2 (c) 3 (d) 8

6. The system of linear equations

$x + y + z = 2$

$2x + y - z = 3$

$3x + 2y + kz = 4$ has a unique solution if

(a) $k \neq 0$ (b) $-1 < k < 1$ (c) $-2 < k < 2$ (d) $k = 0$

7. The value of the determinant $\begin{vmatrix} ka & k^2 + a^2 & 1 \\ kb & k^2 + b^2 & 1 \\ kc & k^2 + c^2 & 1 \end{vmatrix}$ is

(a) $k(a + b)(b + c)(c + a)$ (b) $k\,abc(a^2 + b^2 + c^2)$

(c) $k(a-b)(b-c)(c-a)$ (d) $k(a+b-c)(b+c-a)(c+a-b)$

8. If the system of equations $x + y = 3$, $y + z = 5$, $z + x = 4$, $x + y + kz = 6$ is consistant, then k equals

(a) -1 (b) 1 (c) 2 (d) 0

9. If $x = -9$ is a root of the equation $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$ then the other two roots are

(a) (3, 7) (b) (2, 7) (c) (3, 6) (d) (2, 6)

10. The value of θ lying between $\theta = 0$ and $\theta = \pi/2$ and satisfying the equation

$$\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0, \text{ are}$$

(a) $\frac{3\pi}{24}$ (b) $\frac{5\pi}{24}$ (c) $\frac{11\pi}{24}$ (d) $\frac{\pi}{24}$

11. If l, m, n are the p^{th}, q^{th}, r^{th} terms of a G.P., then $\begin{vmatrix} \log l & p & 1 \\ \log m & q & 1 \\ \log n & r & 1 \end{vmatrix} =$

(a) $p\,q\,r$ (b) $1 + m + n$ (c) 0 (d) none of these

12. If $\begin{vmatrix} 1 & 2 & k \\ k & 4 & 5 \\ 5 & 6 & 7 \end{vmatrix} = 0$, then the values of k are

(a) $-3, 3$ (b) $-3, -\frac{8}{3}$ (c) $3, \frac{8}{3}$ (d) none of these

13. If $Dr = \begin{vmatrix} r & 1 & \frac{n(n+1)}{2} \\ 2r-1 & 4 & n^2 \\ 2^{r-1} & 5 & 2^n - 1 \end{vmatrix}$, then the value of $\sum_{r=0}^{n} Dr$ is

(a) 0 (b) 1 (c) $\frac{n(n+1)(2n+1)}{6}$ (d) none of these

14. $f(x) = \begin{vmatrix} x^3 & x^4 & 3x^2 \\ 1 & -6 & 4 \\ p & p^2 & p^3 \end{vmatrix}$ Here p is a constant then $\frac{d^3 f(x)}{dx^3}$ is

(a) proportional to x^2 (b) proportional to x (c) proportional to x^3 (d) a constant

AIEEE/JEE-MAINS QUESTIONS

1. If $a^2 + b^2 + c^2 = -2$ and $f(x) = \begin{vmatrix} 1+a^2x & (1+b^2)x & (1+c^2)x \\ (1+a^2)x & 1+b^2x & (1+c^2)x \\ (1+a^2)x & (1+b^2)x & 1+c^2x \end{vmatrix}$, then $f(x)$ is a polynomial of degree

(a) 3 (b) 2 (c) 1 (d) 0

2. If $a_1, a_2, a_3, \ldots, a_n$ are in G.P., then the determinant $\Delta = \begin{vmatrix} \log a_n & \log a_{n+1} & \log a_{n+2} \\ \log a_{n+3} & \log a_{n+4} & \log a_{n+5} \\ \log a_{n+6} & \log a_{n+7} & \log a_{n+8} \end{vmatrix}$ is equal to

(a) 4 (b) 2 (c) 1 (d) 0

3. Let $A = \begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix}$. If $|A^2| = 25$, then $|\alpha|$ equals

(a) 1/5 (b) 5 (c) 15^2 (d) 1

4. If the system of linear equations

$x + 2ay + az = 0;\ x + 3by + bz\ \&\ x + 4cy + cz = 0$

has a non-zero solution, then a, b, c

(a) satisfy $a + 2b + 3c = 0$ (b) are in A. P.

(c) are in G.P. (d) are in H. P.

5. If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$ and vectors $(1, a, a^2)$, $(1, b, b^2)$ and $(1, c, c^2)$ are non-coplanar, then the product abc equals

(a) 0 (b) 2 (c) –1 (d) 1

6. l, m, n are the p^{th}, q^{th} and r^{th} terms of a G. P. all positive, then $\begin{vmatrix} \log l & p & 1 \\ \log m & q & 1 \\ \log n & r & 1 \end{vmatrix}$ equals

(a) –1 (b) 2 (c) 1 (d) 0

7. If $a > 0$ and discriminant of $ax^2 + 2bx + c$ is –ve, then $\begin{vmatrix} a & b & ax+b \\ b & c & bx+c \\ ax+b & bx+c & 0 \end{vmatrix}$ is equal to

(c) negative

(d) 0

8. If $D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$ for $x \neq 0$, $y \neq 0$ then D is

(a) divisible by x but not y

(b) divisible by y but not x

(c) divisible by neither x nor y

(d) divisible by both x and y

9. Let A be a square matrix all of whose entries are integers. Then which one of the following is true?

(a) If det $A \neq \pm 1$, then A^{-1} exists and all its entries are non-integers

(b) If det $A = \pm 1$, then A^{-1} exists and all its entries are integers

(c) If det $A = \pm 1$, then A^{-1} need not exist

(d) If det $A = \pm 1$, then A^{-1} exists but all its entries are not necessarily integers

10. Let a, b, c be such that $b(a + c) \neq 0$. If

$$\begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ (-1)^{n+2}a & (-1)^{n+1}b & (-1)^{n}c \end{vmatrix} = 0$$, then the value of n is

(a) any even integer

(b) any odd integer

(c) any integer

(d) zero

11. Let A be a 2 × 2 matrix

Statement - 1 : adj (adj A) = A

Statement - 2 : |adj A| = |A|

(a) Statement-1 and Statement-2 is true; Statement-2 is ***not*** a correct explanation for Statement-1.

(b) Statement-1 is true, Statement-2 is false.

(c) Statement-1 is false, Statement-2 is true.

(d) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

ANSWERS

Basic Level Assignment

1. $x = 0, y = 0$ 2. 0 3. $(x + y + z)^3$ 4. 1

5. 0 8. $x = -1, y = 2, z = 1$ 9. $x = 3k, y = k, z = 3k$ 10. 10

11. $k = \frac{33}{2}$, $x = t$, $y = \frac{-2}{15}t$, $z = \frac{2t}{5}$, where t is an arbitrary non-zero number.

14. $$\begin{vmatrix} 1 & 0 & 2x \\ x+2 & 2x+3 & x \\ x^2 & x^3+1 & 2x^4+1 \end{vmatrix} + \begin{vmatrix} x & 1 & x^2 \\ 1 & 2 & 1 \\ x^2 & x^3+1 & 2x^4+1 \end{vmatrix} + \begin{vmatrix} x & 1 & x^2 \\ x+2 & 2x+3 & x \\ 2x & 3x^2 & 8x^3 \end{vmatrix}$$

16. (i) $b = 3, a \neq 2$; (ii) $a \neq 2, b \neq 3$; (iii) $a = 2$

Advanced Level Assignment

5. $\frac{xyz}{12}(x - y)(y - z)(z - x)$ 6. $\frac{(b^2 - 4ac)}{a^4}(a+b+c)^2$

7. 2 10. $\frac{4d^4}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}$

14. $9 \sin x \cos^2 x$ 19. $a = \frac{1}{4}, b = -\frac{5}{4}, f(x) = \frac{x^2}{4} - \frac{5}{4}x + 2$

20. 0 and independent of a.

Objective Assignment

1.	(d)	2.	(b)	3.	(a)	4.	(a)	5.	(c)
6.	(d)	7.	(d)	8.	(a)	9.	(b)	10	(a)
11.	(a)	12.	(d)	13.	(a)	14.	(d)	15.	(d)
16.	(b)	17.	(b)	18.	(a)	19.	(a)	20.	(b)
21.	(c)	22.	(b)	23.	(c)	24.	(d)	25.	(a)
26.	(d)	27.	(b)	28.	(d)	29.	(d)	30.	(c)
31.	(a,c)	32.	(b,d)	33.	(a,c)	34.	(a,b)	35.	(b,c)
36.	(a,b,c)	37.	(a,c)	38.	(a,c)	39.	(a,b)	40.	(a,d)

1.	(d)	2.	(a)	3.	(a)	4.	(a)	5.	(b)
6.	(c)	7.	(a)						
8.	A-(q); B-(s); C-(r); D-(p)			9.	A-(s); B-(p); C-(q); D-(s)				
10.	(6)	11.	(9)	12.	(3)	13.	(3)	14.	(1)
15.	(2)	16.	(4)	17.	(2)	18.	(3)	19.	(4)

Previous Year Questions

IIT-JEE/JEE-Advance

1.	(b)	2.	(b)	3.	(d)	4.	(a)	5.	(b)
6.	(c)	7.	(d)	8.	(b)	9.	(a)	10.	(7)
11.	(4)	12.	(a,d)						

DCE

1.	(a)	2.	(b)	3.	(d)	4.	(d)	5.	(b)
6.	(a)	7.	(c)	8.	(b)	9.	(b)	10.	(c)
11.	(c)	12.	(c)	13.	(a)	14.	(c)		

Mains questons

1.	(b)	2.	(d)	3.	(a)	4.	(d)	5.	(c)
6.	(d)	7.	(c)	8.	(d)	9.	(b)	10.	(b)
11.	(a)								