SOLVED SUBJECTIVE EXAMPLES

Example 1 :

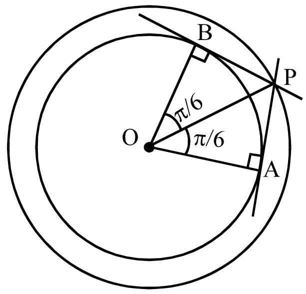
Find the locus of the points of intersection of the tangents to the circle $x = r\cos\theta$, $y = r\sin\theta$ at points whose parametric angles differ by $\pi/3$.

Solution :

All such points P satisfying the given condition will be equidistant from the origin O (see fig.).

Hence the locus of P will be a circle centred at the origin, having radius equal to

$$OP = \frac{r}{\cos\left(\frac{\pi}{6}\right)} = \frac{2r}{\sqrt{3}}$$



Therefore, equation of the required locus is $x^2 + y^2 = \frac{4}{3}r^2$.

Example 2 :

If $-3l^2 - 6l - 1 + 6m^2 = 0$, find the equation of the circle for which $lx + my + 1 = 0$ is a tangent.

Solution :

The given expression can be written as

$$6(l^2 + m^2) = 9l^2 + 6l + 1$$

i.e. $$\frac{3l+1}{\sqrt{l^2+m^2}} = \sqrt{6}\,.$$

From this expression we can infer that the perpendicular distance of the point (3, 0) from the line $lx + my + 1 = 0$ is $\sqrt{6}$.

This proves that the given line is a tangent to the circle $(x-3)^2 + y^2 = 6$.

Example 3 :

Prove that $x^2 + y^2 = a^2$ and $(x-2a)^2 + y^2 = a^2$ are two equal circles touching each other. Find the equation of circle (or circles) of the same radius touching both the circles.

Solution :

Given circles are

$$x^2 + y^2 = a^2 \qquad ....(1)$$

and $$(x-2a)^2 + y^2 = a^2 \qquad ....(2)$$

Let A and B be the centres and r_1 and r_2 the radii of the circles (1) and (2) respectively. Then

$$A \equiv (0, 0),\ B \equiv (2a, 0),\ r_1 = a,\ r_2 = a$$

Now $AB = \sqrt{(0-2a)^2 + 0^2} = 2a = r_1 + r_2$

Hence the two circles touch each other extenally.

Let the equation of the circle having same radius 'a' and touching the circles (1) and (2) be

$$(x-\alpha)^2 + (y-\beta)^2 = a^2 \qquad ....(3)$$

Its centre C is (α, β) and radius $r_3 = a$

Since circle (3) touches the circle (1),

$AC = r_1 + r_3 = 2a$. [Here $AC \neq |r_1 - r_3|$ as $r_1 - r_3 = a - a = 0$]

$\Rightarrow \quad AC^2 = 4a^2 \Rightarrow a^2 + \beta^2 = 4\alpha^2 \qquad ....(4)$

Again since circle (3) touches the circle (2)

$BC = r_2 + r_3 \Rightarrow BC^2 = (r_2) \Rightarrow (2a-\alpha)^2 + \beta^2 = (a+a)^2 \Rightarrow \alpha^2 + \beta^2 - 4a\,\alpha = 0$

$\Rightarrow \quad 4a^2 - 4a\,\alpha = 0$ [from (4)]

$\Rightarrow \quad \alpha = a$ and from (4), $\alpha = 0$ (4), we have $\beta = \pm\sqrt{3}\,a$.

Hence, the required circles are $(x-a)^2 + (y \mp a\sqrt{3})^2 = a^2$ or $x^2 + y^2 - 2ax \mp 2\sqrt{3}\,ay + 3a^2 = 0$

Example 4 :

If the curves $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and $Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$ intersect at four concyclic points, prove that $\frac{(a-b)}{h} = \frac{(A-B)}{H}$.

Solution :

Equation of a curve passing through the intersection points of the given curves can be written as

$$(ax^2 + 2hxy + by^2 + 2gx + 2fy + c) + \lambda(Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C) = 0 \qquad ....(1)$$

If this curve must be a circle, then coeff. of x^2 = coeff. of y^2

i.e. $\quad (a + \lambda A) = (b + \lambda B)$ gives $\lambda = \frac{b-a}{A-B} \qquad ....(2)$

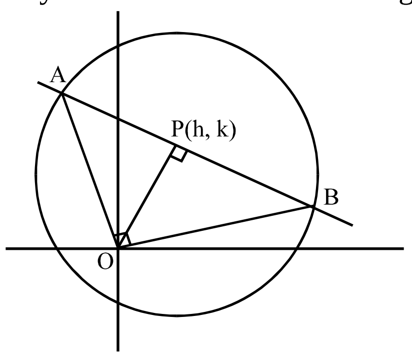
and coeff. of $xy = 0$ i.e. $2(h + \lambda H) = 0$ given $\lambda = -\frac{h}{H} \qquad ....(3)$

Equating the two values of λ, we get the desired result.

Example 5 :

Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be a given circle. Find the locus of the foot of the perpendicular drawn from the origin upon any chord of S which subtends right angle at the origin.

Solution :



AB is a variable chord such that = $\angle AOB = \frac{\pi}{2}$.

Let P(h, k) be the foot of the perpendicular drawn from origin upon AB. Equation of the chord AB is $y - k = \frac{-h}{k}(x - h)$

i.e. $hx + ky = h^2 + k^2$ ....(1)

Equation of the pair of straight lines passingh through the origin and the intersection point of the given circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad ....(2)$$

and the variable chord AB is

$$x^2 + y^2 + 2(gx + fy)\left(\frac{hx + ky}{h^2 + k^2}\right) + c\left(\frac{hx + ky}{h^2 + k^2}\right)^2 = 0 \qquad ....(3)$$

If equation (3) must represent a pair of perpendicular lines, then we have

coeff. of x^2 + coeff. of $y^2 = 0$

i.e. $$\left(1 + \frac{2gh}{h^2 + k^2} + \frac{ch^2}{(h^2 + k^2)^2}\right) + \left(1 + \frac{2fk}{h^2 + k^2} + \frac{ck^2}{(h^2 + k^2)^2}\right) = 0.$$

Putting (x, y) in place of (h, k) gives the equation of the required locus as

$$x^2 + y^2 + gx + fy + \frac{c}{2} = 0.$$

***Example* 6 :**

The line Ax + By + C = 0 cuts the circle $x^2 + y^2 + gx + fy + c = 0$ at P and Q. The line $A'x + B'y + C' = 0$ cuts the circle $x^2 + y^2 + g'x + f'y + c' = 0$ at R and S. If P, Q, R and S are concyclic, show that $\begin{vmatrix} g - g' & f - f' & c - c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0.$

Solution :

Equation of a circle through P and Q is $x^2 + y^2 + gx + fy + c + \lambda(Ax + By + C) = 0$

i.e. $x^2 + y^2 + (g + \lambda A)x + (f + \lambda B)y + (c + \lambda C) = 0$ ....(1)

and equation of a circle through R and S is $x^2 + y^2 + g'x + f'y + c' + \mu(A'x + B'y + C') = 0$

$x^2 + y^2 + (g' + \mu A')y + (f' + \mu B') + (c' + \mu C') = 0$ ....(2)

If P, Q, R and S are concyclic points, then equations (1) and (2) must represent the same circle.

Equating the ratio of the coefficients, we have $1 = \frac{g + \lambda A}{g' + \mu A'} = \frac{f + \lambda B}{f' + \mu B'} = \frac{c + \lambda C}{c' + \mu C'}$

i.e. $\lambda A - \mu A' + g - g' = 0$ ....(3)

$\lambda B - \mu B' + f - f' = 0$ ....(4)

and $\lambda C - \mu C' + c - c' = 0$ ....(5)

Eliminating λ and μ from equation (3), (4) and (5), we have $\begin{vmatrix} A & -A' & g-g' \\ B & -B' & f-f' \\ C & -C' & c-c' \end{vmatrix} = 0$

or $\begin{vmatrix} g-g' & f-f' & c-c' \\ A & B & C \\ A' & B' & C' \end{vmatrix}$ [interchanging rows by columns and then interchanging the second and the third row]

Aliter :

Let the given circles be $S_1 \equiv x^2 + y^2 + gx + fy + c = 0$ ......(1)

and $S_2 \equiv x^2 + y^2 + g'x + f'y + c' = 0$ ....(2)

If S be the required circle, then according to the given condition

$Ax + By + C = 0$ is the radical axis of S_1, S and $A'x + B'y + C' = 0$ is the radical axis of S_2, S while $(g - g')x + (f - f')y + (c - c') = 0$ is the radical axis of S_1, S_2.

Since the radical axes of three circles taken in pairs are concurrent, therefore, we have

$$\begin{vmatrix} g-g' & f-f' & c-c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0$$ which is the desired result.

Example 7 :

Circles are drawn passingh through the origin O to intersect the coordinate axes at point P and Q such that m . OP + n . OQ is a constant. Show that the circles pass through a fixed point.

Solution :

Equation of a circle passing through the origin and having X and Y intercepts equal to a and b respctively is $x^2 + y^2 - ax - by = 0$ ....(1)

According to the given condition, we have

$ma + nb = k$ (constant) i.e. $b = \dfrac{k - ma}{n}$ ......(2)

Putting the above value of b in equation (1) , we have, $x^2 + y^2 - ax - \left(\dfrac{k-ma}{n}\right) y = 0$

i.e. $\{n(x^2 + y^2) - ky\} - a(nx - my) = 0$

which represents the equation of a family of circles passing through the intersection points of the circle $n(x^2 + y^2) - ky = 0$ ....(3)

and the line $nx - my = 0$ ....(4)

Solving equation (3) and (4), gives the coordinates of the fixed point as $\left(\dfrac{mk}{m^2+n^2}, \dfrac{nk}{m^2+n^2}\right)$.

Example 8 :

P(p, q) is a point on a circle passing through the origin and centred at $C\left(\frac{p}{2}, \frac{q}{2}\right)$. If two distinct chords can be drawn from P such that these chords are bisected by the X-axis, then show that $p^2 > 8q^2$.

Solution :

It can be seen that the given points P(p, q), $C\left(\frac{p}{2}, \frac{q}{2}\right)$ and the origin are collinear which implies that line OP where O is the origin is a diameter of the given circle. Therefore, equation of the given circle is

$$x(x - p) + y(y - q) = 0$$

i.e. $\quad x^2 + y^2 - px - qy = 0 \quad$ ....(1)

Let M(a, 0) be the mid-point of a chord AP (see fig.). Then, we have

$$CM \perp AP$$

i.e. $\quad$ slope of CM × slope of AP = – 1 $\quad \dfrac{\frac{q}{2}}{\frac{p}{2} - a} \times \dfrac{q}{p - a} = -1$

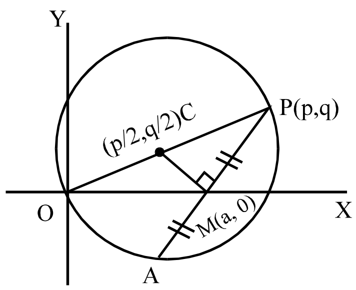
i.e. $\quad q^2 + (p - 2a)(p - a) = 0$

i.e. $\quad 2a^2 - 3pa + p^2 + q^2 = 0 \quad$ .....(2)

Equation (2) which is a quadratic equation in a shows that there will be two real and distinct values of a if the discriminant is > 0

i.e. if $(3p)^2 - 4 \times 2(p^2 + q^2) > 0$

i.e. if $p^2 > 8q^2$



which is the desired result.

Aliter. Equation of the given circle is

$x^2 + y^2 - px - qy = 0 \quad$ ....(1)

Equation of any line through P(p, q) can be written as

$y - q = m(x - p)$ (where m is a variable)

i.e. $\quad x = \dfrac{y + (mp - q)}{m} \quad$ ......(2)

putting the value of x from equation (2) in equation (1) will give the ordinate of the intersection points of the line and the given circle as

$$\left\{\frac{y + (mp - q)}{m}\right\}^2 + y^2 - p\left\{\frac{y + (mp - q)}{m}\right\} - qy = 0$$

i.e. $\{y + (mp - q)\}^2 + m^2y^2 - mp\{y + (mp - q)\} - m^2qy = 0$

i.e. $(1 + m^2)y^2 + \{2(mp - q) - mp - m^2q\}y + (mp - q)^2 - mp(mp - q) = 0$

i.e. $(1 + m^2)y^2 + (pm - 2q - qm^2)y - q(mp - q) = 0$ ....(3)

The above equation gives the Y coordinates of the intersection points of the chord and the given circle. According to the given condition, the mid-point of this intercept lies on the X-axis, therefore we have sum of the roots of equation (3) = 0

i.e. $pm - 2q - qm^2 = 0$

i.e. $qm^2 - pm + 2q = 0$ ....(4)

The above equation shows that there will be two real and distinct values of m if $p^2 > 8q^2$ which is the desired result.

***Example* 9 :**

Prove that the square of the tangent that can be drawn from any point on one circle to another circle is equal to twice the product of perpendicular distance of the point from the radical axis of two circles and distance between their centres.

Solution :

Let us choose the circles, as $S_1 \equiv x^2 + y^2 - a^2 = 0$ ....(1)

and $S_2 \equiv (x - b)^2 + y^2 - c^2 = 0$ ....(2)

Let P(a cosθ, asinθ) be any point on circle S_1. The length of the tangent from P to circle S_2, is given by $PT^2 = S_2(a\cos\theta, a\sin\theta) = (a\cos\theta - b)^2 + (a\sin\theta)^2 - c^2 = a^2 + b^2 - c^2 - 2ab\cos\theta$

The distance between the centres of S_1 and S_2, is

$C_1C_2 = b$

The radical axis of S_1 and S_2, is $2bx - a^2 - b^2 + c^2 = 0$

[equation (1) – equation (2)]

The perpendicular distance of P from the radical axis, is

$$PM = \frac{|2b(a\cos\theta) - a^2 - b^2 + c^2|}{2b}$$

Now, we have

$2.\ PM.\ C_1C_2 = 2b.\ \dfrac{|2ab\cos\theta - a^2 - b^2 + c^2|}{2b} = |a^2 + b^2 - c^2 - 2ab\cos\theta| = PT^2$ which proves the desired result.

***Example* 10 :**

Consider a family of circles passing through the intersection point of the lines $\sqrt{3}(y - 1) = x - 1$ and $y - 1 = \sqrt{3}(x - 1)$ and having its centre on the acute angle bisector of the given lines. Show that the common chords of each member of the family and the circle $x^2 + y^2 + 4x - 6y + 5 = 0$ are concurrent. Find the point of concurrency.

Solution :

The given lines $\sqrt{3}(y - 1) = x - 1$ ....(1)

$y - 1 = \sqrt{3}(x - 1)$ ......(2)

intersect at the point (1, 1).

Rewriting the equation of the given lines such that their constant terms are both positive, we have

$x - \sqrt{3}\,y + \sqrt{3} - 1 = 0$ ......(3)

and $-\sqrt{3}\,x + y + \sqrt{3} - 1 = 0$ ......(4)

Here, we have

(product of coeff.'s of x) + (product of coeff.'s of y) = $-\sqrt{3} - \sqrt{3}$ = – ve quantity

which implies that the acute angle between the given lines contains the origin.

Therefore, equation of the acute angle bisector of the given lines is

$$\frac{x - \sqrt{3}y + \sqrt{3} - 1}{2} = + \frac{-\sqrt{3}x + y + \sqrt{3} - 1}{2}$$

i.e. $y = x$

Any point on the above bisector can be chosen as (α, α) and equation of any circle passing through (1, 1) and having centre at (α, α) is

$(x - \alpha)^2 + (y - \alpha)^2 = (1 - \alpha)^2 + (1 - \alpha)^2$

i.e. $x^2 + y^2 - 2\alpha x - 2\alpha y + 4\alpha - 2 = 0$ ....(6)

The common chord of the given circle

$x^2 + y^2 + 4x - 6y + 5 = 0$ ....(7)

and the circle represented by equation (6) is

$(4 + 2\alpha)x + (2\alpha - 6)y + (7 - 4\alpha) = 0$

i.e. $(4x - 6y + 7) + 2\alpha(x + y - 2) = 0$ ....(8)

which represents a family of straight lines passingh through the intersection point of the lines

$4x - 6y + 7 = 0$ ....(9)

and $x + y - 2 = 0$ ....(10)

Solving equation (9), (10) gives the coordinates of the fixed point as $\left(\frac{1}{2}, \frac{3}{2}\right)$.

***Example* 11 :**

Find the range of values of λ for which the variable line $3x + 4y - \lambda = 0$ lies between the circles $x^2 + y^2 - 2x - 2y + 1 = 0$ and $x^2 + y^2 - 18x - 2y + 78 = 0$ without intercepting a chord on either circle.

***Solution* :**

The given circle

$S_1 \equiv x^2 + y^2 - 2x - 2y + 1 = 0$ ....(1)

has centre $C_1 \equiv (1, 1)$ and radius $r_1 = 1$

The other given circle

$S_2 \equiv x^2 + y^2 - 18x - 2y + 78 = 0$ ....(2)

has centre $C_2 \equiv (9, 1)$ and radius $r_2 = 2$.

According to the required condition, we have

$C_1M_1 \geq r_1$

i.e. $\frac{|3+4-\lambda|}{\sqrt{3^2+4^2}} \geq 1$

i.e. $(\lambda - 7) \geq 5$ [$\because$ C_1 lies below the line $\therefore$ $(7 - \lambda)$ is a –ve quantity]

i.e. $\lambda \geq 12$i.e.

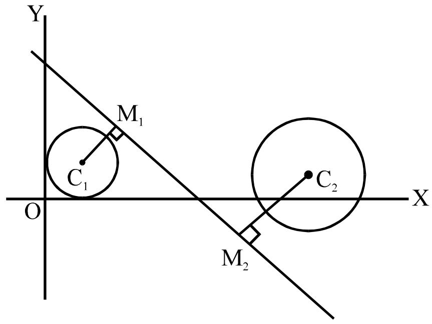
$\frac{|27+4-\lambda|}{\sqrt{3^2+4^2}} \geq 2$

i.e. $(31 - \lambda) \geq 10$ [$\because$ C_1 lies below the line

$\therefore$ $(31 - \lambda)$ is a +ve quantity]

i.e. $\lambda \leq 21$

Hence, the permissible values of λ are $12 \leq \lambda \leq 21$.



Example 12 :

Point P having integral coordinates lies on $x^2 + y^2 = 1$ and $x^2 + y^2 + 2x + 4y + 1 = 0$. A chord through P meets the two circles at A and B. Find the equation of the chord PAB if PA and PB subtend equal angles at the centres of the respective circles.

Solution :

Equation of the given circles are $S_1 \equiv x^2 + y^2 - 1 = 0$ ....(1)

and $S_2 \equiv x^2 + y^2 + 2x + 4y + 1 = 0$ ....(2)

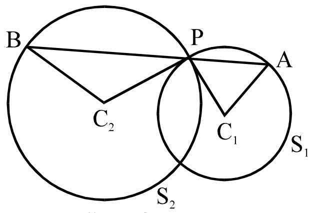
Subtracting equation (2) from equation (1), we have

$x = -(2y + 1)$ ....(3)

Putting in equation (1), we have $(2y + 1)^2 + y^2 = 1$

i.e. $5y^2 + 4y = 1$ gives $y = 0, -4/5$

and the corresponding values of $x = -1, 3/5$.



radius of S_1 = 1
radius of S_2 = 2

Thus, the intersection point of circles S_1 and S_2, having intergral coordinates, is

$P \equiv (-1, 0)$.

From the fig., we can see that if PA and PB subtend equal angles at C_1 and C_2 respectively, then $PA : PB = C_1A : C_2B = 1 : 2$

Equation of a line through P can be chosen as

$y = m(x + 1)$

Soving equations (1) and (4) for the intersection point $A(\alpha, \beta)$ (say), we have

$x^2 + m^2(x + 1)^2 = 1$

i.e. $(1 + m^2)x^2 + 2m^2x + (m^2 - 1) = 0$

whose one root is $x = -1$ since one of the intersection point is $P(-1, 0)$.

Thus, we have $-1.\ \alpha = \frac{m^2-1}{1+m^2}$ gives $\alpha = \frac{1-m^2}{1+m^2}$

Soving equations (2) and (4) for the intersection point B(λ, δ) (say), we have

$$x^2 + m^2(x + 1)^2 + 2x + 4m(x + 1) + 1 = 0$$

i.e. $(1 + m^2)x^2 + (2m^2 + 4m + 2)x + (m^2 + 4m + 1) = 0$

whose one root is x = – 1 since one of the intersection point is P(–1, 0).

Thus, we have $-1.\ \lambda = \frac{m^2+4m+1}{1+m^2}$ gives $\lambda = -\left(\frac{m^2+4m+1}{1+m^2}\right)$

Now, using the condition PA : PB = 1 : 2, we have

$2\alpha + \lambda = -3$

i.e. $2(1 - m^2) - (m^2 + 4m + 1) = -3(1 + m^2)$

gives $m = \frac{3}{4}$.

Hence, equation of the required chord is $y = \frac{3}{4}(x + 1)$.

***Example* 13 :**

Curves $ax^2 + 2hxy + by^2 - 2gx - 2fy + c = 0$ and $a'x^2 - 2hxy + (a' + a - b)y^2 - 2g'x - 2f'y + c = 0$ intersect at four concyclic point A, B, C and D. If P is the point $\left(\frac{g'+g}{a'+a}, \frac{f'+f}{a'+a}\right)$ prove that $PA^2 + PB^2 + PC^2 = 3PD^2$.

Solution :

Equation of a curve passing through the intersection points of the given curves

$$ax^2 + 2hxy + by^2 - 2gx - 2fy + c = 0 \qquad \text{........(1)}$$

and $a'x^2 - 2hxy + (a' + a - b)y^2 - 2g'x - 2f'y + c = 0$ ....(2)

can be written as $\{a'x^2 - 2hxy + (a' + a - b)y^2 - 2g'x - 2f'y + c\}$

$$+ \lambda\{ax^2 + 2hxy + by^2 - 2gx - 2fy + c\} = 0$$

i.e. $(a' + \lambda a)x^2 + 2h(\lambda - 1)xy + (a' + a - b + \lambda b)y^2$

$$- 2(g' + \lambda g)x - 2(f' + \lambda f)y + (1 + \lambda)c = 0 \qquad \text{........(3)}$$

According to the given condition equation (3) must represent a circle, therefore, we have

coeff. of x^2 = coeff. of y^2

i.e. $a' + \lambda a = a' + a - b + \lambda b$

i.e. $\lambda(a - b) = a - b$

gives $\lambda = 1$

and coeff. of xy = 0

i.e. $\lambda - 1 = 0$

gives $\lambda = 1$.

The identical values prove that the curve is a circle.

Putting the above value of λ in equation (3) gives the equation of the circle passing through the intersection points of the curves represented by equations (1) and (2) as $(a' + a)(x^2 + y^2) - 2(g' + g)x - 2(f' + f)y + 2c = 0$

which has its centre at the point $\left(\frac{g'+g}{a'+a}, \frac{f'+f}{a'+a}\right)$.

We can see that the coordinates of the given point P is the same as the centre of the circle passing through the points A, B, C and D. Therefore, we have $PA^2 = PB^2 = PC^2 = PD^2$ = radius of the circle which gives the desired result $PA^2 + PB^2 + PC^2 = 3PD^2$.

Example 14 :

A is one of the points of intersection of two given circles. A variable line through A meets the two circles again at point P and Q. Show that the locus of the mid-point of P and Q is also a circle passing through A.

Solution :

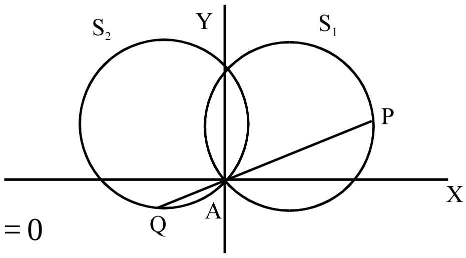
Let us choose the intersection point A as the origin and the radical axis of the circles, as the Y-axis (see fig.). Then the equation of the circles can be chosen as

$S_1 \equiv x^2 + y^2 - 2g_1x - 2fy = 0$ ......(1)

and $S_2 \equiv x^2 + y^2 - 2g_2x - 2fy = 0$ ....(2)

Equation of a variable line through A can be written as $y = mx$.

Putting in equation (1), we have



$$x^2(1 + m^2) - 2(g_1 + mf)x = 0$$

gives $x = 0, \frac{2(g_1 + mf)}{1+m^2}$

Putting in equation (2), we have $x^2(1 + m^2) - 2(g_2 + mf)x = 0$

gives $x = 0, \frac{2(g_2 + mf)}{1+m^2}$

Thus, we have $P \equiv (x_1, mx_1)$ where $x_1 = \frac{2(g_1 + mf)}{1+m^2}$

and $Q \equiv (x_2, mx_2)$ where $x_2 = \frac{2(g_2 + mf)}{1+m^2}$

If M(h, k) be the mid-point of PQ, then $2h = x_1 + x_2$

i.e. $h(1 + m^2) = g_1 + g_2 + 2mf$ ....(3)

and $2k = m(x_1 + x_2)$

i.e. $k(1 + m^2) = m(g_1 + g_2 + 2mf)$ ......(4)

Dividing equation (4) by equation (3), we have $m = \frac{k}{h}$

Putting the above value of m in equation (3), we have

$$h\left(1+\frac{k^2}{h^2}\right) = g_1 + g_2 + \frac{2kf}{h} \text{ i.e. } h^2 + k^2 = (g_1 + g_2)h + 2fk$$

Putting (x, y) in place of (h, k) gives the equation of the required locus, as

$$x^2 + y^2 - (g_1 + g_2)x - 2fy = 0$$ which is a circle passing through A(0, 0).

Example 15 :

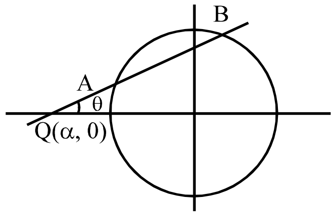
Q is a fixed point and S is a fixed circle. A variable chord through Q meets the circle S at point A and B. Find the locus of a point P on this chord such that QA, QP, QB are in

(A) arithmetic progression

(B) geometric progression

(C) harmonic progression

Solution :



Let us choose the line joining Q and the centre of the circle S as the X-axis and the centre of the circle as the origin (see fig.).

Let the coordinates of the fixed point Q be (α, 0) and the equation of the fixed circle S be

$x^2 + y^2 = a^2$ ....(1)

Let θ be the inclinationof a variable line through Q. The coordinates of any point on this line can be chosen as $(\alpha + r\cos\theta, r\sin\theta)$. If this point also lies on the circle S, then putting the above coordinates in equation (1), we have

$$(\alpha + r\cos\theta)^2 + (r\sin\theta)^2 = a^2$$

i.e. $r^2 + (2\alpha\cos\theta)\, r + \alpha^2 - a^2 = 0$ ....(2)

The roots of the above equation , say r_1, r_2 are the distance QA and QB. Thus, we have

$$QA + QB = r_1 + r_2 = -2\alpha\cos\theta$$

$$QA.\ QB = r_1 r_2 = \alpha^2 - a^2$$

Let P(h, k) be the point whose locus is to be found. If the distance QP is denoted by r, then we have $h = \alpha + r\cos\theta$, $k = r\sin\theta$

(A) If QP is the A.M. of QA and QB. then $r = \frac{r_1 + r_2}{2} = -\alpha\cos\theta$ [from equation (2)]

Thus, we have $h = \alpha - \alpha\cos^2\theta = \alpha\sin^2\theta$ ....(3)

$k = -\alpha\cos\theta\sin\theta$ ....(4)

Now, we have $\frac{k^2}{h} = \alpha\cos^2\theta$ ....(5)

Adding equations (3) and (4), we have $h^2 + k^2 = \alpha h$

Hence, the required locus is $x^2 + y^2 - \alpha x = 0$ which is a circle.

(B) If QP is the G.M. of QA and QB. then

$r = \sqrt{r_1 r_2} = \sqrt{\alpha^2 - a^2}$ [from equation 2]

Thus , we have $h = \alpha - \sqrt{\alpha^2 - a^2}\cos\theta$, $k = \sqrt{\alpha^2 - a^2}\sin\theta$

Eliminating θ, we have $(h - \alpha)^2 + k^2 = \alpha^2 - a^2$

i.e. $h^2 + k^2 - 2\alpha h + a^2 = 0$

Hence, the required locus is $x^2 + y^2 - 2\alpha x + a^2 = 0$ which is a circle.

(C) If QP is the H.M. of QA and QB, then $r = \dfrac{2r_1r_2}{r_1 + r_2} = \dfrac{\alpha^2 - a^2}{-\alpha\cos\theta}$

Thus, we have $h = \alpha - \dfrac{\alpha^2 - a^2}{\alpha}$, $k = \left(\dfrac{\alpha^2 - a^2}{-\alpha\cos\theta}\right)\sin\theta$

From the first equation above, θ is eliminated. $\alpha h = a^2$

Hence, the required locus is $\alpha x = a^2$.

SOLVED OBJECTIVE EXAMPLES

***Example* 1 :**

Tangents are drawn to the circle $x^2 + y^2 = 50$ from a point 'P' lying on the x-axis. These tangents meet the y-axis at points 'P_1' and 'P_2'. Possible coordinates of 'P' so that area of triangle PP_1P_2 is minimum, is/are

(A) (10, 0)

(B) $(10\sqrt{2}, 0)$

(C) (–10, 0)

(D) $(-10\sqrt{2}, 0)$

Solution :

$OP = 5\sqrt{2}\ \sec\theta,$

$OP_1 = 5\sqrt{2}\ \text{cosec}\theta$

area $(\Delta PP_1P_2) = \dfrac{100}{\sin 2\theta}$, area $(\Delta PP_1\ P_2)_{min} = 100$

$\Rightarrow \quad \theta = \pi/4 \Rightarrow OP = 10$

$\Rightarrow \quad P = (10, 0), (-10, 0)$

Hence (A), (C) are correct

***Example* 2 :**

Two circles with radii 'r_1' and 'r_2', $r_1 > r_2 \geq 2$, touch each other externally. If 'θ' be the angle between the direct common tangents, then

(A) $\theta = \sin^{-1}\left(\dfrac{r_1 + r_2}{r_1 - r_2}\right)$

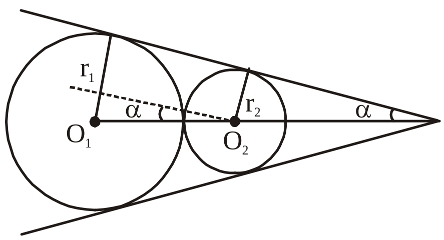
(B) $\theta = 2\sin^{-1}\left(\dfrac{r_1 - r_2}{r_1 + r_2}\right)$

(C) $\theta = \sin^{-1}\left(\dfrac{r_1 - r_2}{r_1 + r_2}\right)$

(D) none of these

Solution:

$\sin\alpha = \dfrac{r_1 - r_2}{r_1 + r_2}$



$\Rightarrow \theta = 2\sin^{-1}\left(\dfrac{r_1 - r_2}{r_1 + r_2}\right)$

Hence (B) is correct

***Example* 3 :**

If the curves $ax^2 + 4xy + 2y^2 + x + y + 5 = 0$ and $ax^2 + 6xy + 5y^2 + 2x + 3y + 8 = 0$ intersect at four concyclic points then the value of a is

(A) 4 (B) – 4

(C) 6 (D) – 6

Solution :

Any second degree curve passing through the intersection of the given curves is

$$ax^2 + 4xy + 2y^2 + x + y + 5 + \lambda (ax^2 + 6xy + 5y^2 + 2x + 3y + 8) = 0$$

If it is a circle, then coefficient of x^2 = coefficient of y^2 and coefficient of xy = 0

$$a (1 + \lambda) = 2 + 5\lambda \text{ and } 4 + 6\lambda = 0$$

$$\Rightarrow \quad a = \frac{2+5\lambda}{1+\lambda} \text{ and } \lambda = -\frac{2}{3} \Rightarrow a = \frac{2-\frac{10}{3}}{1-\frac{2}{3}} = -4.$$ Hence (B) is correct answer.

Example 4 :

The chords of contact of the pair of tangents drawn from each point on the line 2x + y = 4 to the circle $x^2 + y^2 = 1$ pass through a fixed point -

(A) (2, 4) (B) $\left(-\frac{1}{2}, -\frac{1}{4}\right)$

(C) $\left(\frac{1}{2}, \frac{1}{4}\right)$ (D) (–2, – 4)

Solution :

The chord of contact of tangents from (α, β) is

$\alpha x + \beta y = 1$ ....(1)

Hence, (1) passes through $\left(\frac{1}{2}, \frac{1}{4}\right)$. Hence (C) is correct answer.

Example 5 :

Equation of chord AB of circle $x^2 + y^2 = 2$ passing through P(2, 2) such that PB/PA = 3, is given by-

(A) x = 3y (B) x = y

(C) $y - 2 = \sqrt{3}\,(x - 2)$ (D) none of these

Solution :

Any line passing through (2, 2) will be of the form $\frac{y-2}{\sin\theta} = \frac{x-2}{\cos\theta} = r$

When this line cuts the circle $x^2 + y^2 = 2$, $(r\cos\theta + 2)^2 + (2 + r\sin\theta)^2 = 2$

$\Rightarrow \quad r^2 + 4(\sin\theta + \cos\theta) r + 6 = 0 \quad \frac{PB}{PA} = \frac{r_2}{r_1}$, now if $r_1 = \alpha$, $r_2 = 3\alpha$, then $4\alpha = -4(\sin\theta + \cos\theta)$, $3\alpha^2 = 6 \Rightarrow \sin 2\theta = 1 \Rightarrow \theta = \pi/4$.

So required chord will be $y - 2 = 1(x - 2) \Rightarrow y = x$.

Alternative solution

$PA.\ PB = PT^2 = 2^2 - 2 = 6$ ....(1)

$\dfrac{PB}{PA} = 3$ ....(2)

From (1) and (2), we have $PA = \sqrt{2}$, $PB = 3\sqrt{2}$

$\Rightarrow AB = 2\sqrt{2}$. Now diameter of the circle is $2\sqrt{2}$ (as radius is $\sqrt{2}$)

Hence line passes through the centre $\Rightarrow y = x$.

Hence (B) is the correct answer.

Example 6 :

Equation of a circle $S(x, y) = 0$, $(S(2, 3) = 16)$ which touches the line $3x + 4y - 7 = 0$ at $(1, 1)$ is given by

(A) $x^2 + y^2 + x + 2y - 5 = 0$
(B) $x^2 + y^2 + 2x + 2y - 6 = 0$
(C) $x^2 + y^2 + 4x - 6y = 0$
(D) none of these

Solution :

Any circle which touches $3x + 4y - 7 = 0$ at $(1, 1)$ will be of the form

$S(x, y) \equiv (x - 1)^2 + (y - 1)^2 + \lambda(3x + 4y - 7) = 0$

Since $S(2, 3) = 16 \Rightarrow \lambda = 1$, so required circle will be

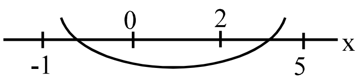
$x^2 + y^2 + x + 2y - 5 = 0$. Hence (A) is the correct answer.

Example 7 :

If $(a, 0)$ is a point on a diameter of the circle $x^2 + y^2 = 4$, then $x^2 - 4x - a^2 = 0$ has

(A) exactly one real root in $(-1, 0]$
(B) Exactly one real root in $[2, 5]$
(C) distinct roots greater than-1
(D) Distinct roots less than 5

Solution :



Since $(a, 0)$ is a point on the diameter of the circle $x^2 + y^2 = 4$,

So maximum value of a^2 is 4

Let $f(x) = x^2 - 4x - a^2$

clearly $f(-1) = 5 - a^2$ is 4

$f(2) = -(a^2 + 4) < 0$

$f(0) = -a^2 < 0$ and $f(5) = 5 - a^2 > 0$, so graph of $f(x)$ will be as shown

Hence (A), (B), (C), (D) are the correct answer.

Example 8 :

If a circle $S(x, y) = 0$ touches at the point $(2, 3)$ of the line $x + y = 5$ and $S(1, 2) = 0$, then radius of such circle.

(A) 2 units
(B) 4 units
(C) $\dfrac{1}{2}$ units
(D) $\dfrac{1}{\sqrt{2}}$ units

Solution :

Desired equation of the circle is

$(x - 2)^2 + (y - 3)^2 + \lambda(x + y - 5) = 0$

$1 + 1 + \lambda\ (1 + 2 - 5) = 0 \Rightarrow \lambda = 1$

$x^2 - 4x + 4 + y^2 - 6y + 9 + x + y - 5 = 0 \Rightarrow x^2 + y^2 - 3x - 5y + 8 = 0$

$$\left(x^2 + \frac{3}{2}\right)^2 + \left(y - \frac{5}{2}\right)^2 = -8 + \frac{25}{4} + \frac{9}{4} = \frac{2}{4} = \frac{1}{2}.$$

Hence (D) is the correct answer.

Example 9 :

If P(2, 8) is an interior point of a circle $x^2 + y^2 - 2x + 4y - p = 0$ which neither touches nor intersects the axes, then set for p is -

(A) $p < -1$

(B) $P < -4$

(C) $p > 96$

(D) ϕ

Solution :

For internal point p(2, 8) $4 + 64 - 4 + 32 - p < 0 \Rightarrow p > 96$ and x intercept = $2\sqrt{1+p}$ therefore

$1 + p < 0 \Rightarrow\ p < -1$ and y intercept = $2\sqrt{4+p}\ \Rightarrow p < -4$

Hence (D) is the correct answer.

Example 10 :

If two circles $(x - 1)^2 + (y - 3)^2 = r^2$ and $x^2 + y^2 - 8x + 2y + 8 = 0$ intersect in two distinct point then

(A) $2 < r < 8$

(B) $r < 2$

(C) $r = 2$

(D) $r > 2$

Solution :

Let d be the distance between the centres of two circles of radii r_1 and r_2.

These circle intersect at two distinct points if $|r_1 - r_2| < d < r_1 + r_2$

Here, the radii of the two circles are r and 3 and distance between the centres is 5.

centres is 5.

Thus, $|r - 3| < 5\ r + 3\ \Rightarrow -2 < r < 8$ and $r > 2 \Rightarrow 2 < r < 8$. Hence (A) is the coorect answer.

Example 11 :

The common chord of $x^2 + y^2 - 4x - 4y = 0$ and $x^2 + y^2 = 16$ subtends at the origin an angle equal to

(A) $\pi/6$

(B) $\pi/4$

(C) $\pi/3$

(D) $\pi/2$

Solution :

The equation of the common chord of the circles $x^2 + y^2 - 4x - 4y = 0$ and $x^2 + y^2 = 16$ is $x + y = 4$ which meets the circle $x^2 + y^2 = 16$ at points A(4, 0) and B(0, 4). Obviously $OA \perp OB$. Hence the common chord AB makes a right angle at the centre of the circle $x^2 + y^2 = 16$.

Hence (D) is the correct answer.

***Example* 12 :**

The number of common tangents that can be drawn to the circle $x^2 + y^2 - 4x - 6y - 3 = 0$ and $x^2 + y^2 + 2x + 2y + 1 = 0$ is

(A) 1 (B) 2 (C) 3 (D) 4

Solution :

The two circles are

$x^2 + y^2 - 4x - 6y - 3 = 0$ and $x^2 + y^2 + 2x + 2y + 1 = 0$

Centre : $C_1 \equiv (2, 3)$, $C_2 \equiv (-1, -1)$ radii : $r_1 = 4$, $r_2 = 1$

We have $C_1 C_2 = 5 = r_1 + r_2$, therefore there are 3 common tangents to the given circles. Hence (C) is the correct answer.

***Example* 13 :**

The tangents drawn from the origin to the circle $x^2 + y^2 - 2rx - 2hy + h^2 = 0$ are perpendicular if

(A) $h = r$ (B) $h = -r$ (C) $r^2 + h^2 = 1$ (D) $r^2 = h^2$

Solution :

The combined equation of the tangents drawn from (0, 0) to $x^2 + y^2 - 2rx - 2hy + h^2 = 0$ is

$$(x^2 + y^2 - 2rx - 2hy + h^2)h^2 = (-rx - hy + h^2)^2$$

This equation represents a pair of perpendicular straight lines if Coeff. of x^2 + coeff. of $y^2 = 0$ i.e.

$2h^2 - r^2 - h^2 = 0 \Rightarrow r^2 = h^2$ or $r = \pm h$. Hence (A), (B), and (D) are correct answers.

***Example* 14 :**

The equation (s) of the tangent at the point (0, 0) to the circle, making intercepts of length 2a and 2b units on the coordinate axes, is (are) -

(A) $ax + by = 0$ (B) $ax - by = 0$ (C) $x = y$ (D) None of these

Solution :

Equation of circle passing through origin and cutting off intercepts 2a and 2b units on the coordinate axes is $x^2 + y^2 \pm 2ax \pm 2by = 0$. Hence (A), (B) are correct answers.

***Example* 15 :**

The slope of the tangent at the point (h, h) of the circle $x^2 + y^2 = a^2$ is -

(A) 0 (B) 1 (C) – 1 (D) depend on h

Solution :

The equation of the tangent at (h, h) to $x^2 + y^2 = a^2$ is $hx + hy = a^2$.

Therefore slope of the tangent $= -h/h = -1$. Hence (C) is the correct answer.