SOLVED SUBJECTIVE EXAMPLES

Example 1:

The lengths of sides of a triangle are three consecutive natural numbers and its largest angle is twice the smallest one. Determine the sides of the triangle.

Solution:

Let the lengths of the sides be n, n + 1, n + 2, where $n \in N$.

From the question, the largest angle opposite to the side n + 2 is 2θ while the smallest angle oposite to the side n is θ.

Now $\cos\theta = \dfrac{(n+1)^2+(n+2)^2-n^2}{2(n+1)(n+2)} = \dfrac{n^2+6n+5}{2(n+1)(n+2)} = \dfrac{(n+1)(n+5)}{2(n+1)(n+2)} = \dfrac{n+5}{2(n+2)}$

and $\cos 2\theta = \dfrac{n^2+(n+1)^2-(n+2)^2}{2n(n+1)} = \dfrac{n^2-2n-3}{2n(n+1)} = \dfrac{(n+1)(n-3)}{2n(n+1)} = \dfrac{n-3}{2n}$

But $\cos 2\theta = 2\cos^2\theta - 1$; so $\dfrac{n-3}{2n} = 2\left\{\dfrac{n+5}{2(n+2)}\right\}^2 - 1$

or $\dfrac{n-3}{2n} = \dfrac{(n+5)^2}{2(n+2)^2} - 1$ or $(n-3)(n+2)^2 = n\{(n+5)^2 - 2(n+2)^2\}$

or $(n-3)(n^2+4n+4) = n(-n^2+2n+17)$

or $n^3+n^2-8n-12 = -n^3+2n^2+17n$

or $(n-4)(2n^2+7n+3) = 0$ $\quad\therefore\quad$ $n = 4$ or $2n^2+7n+3 = 0$.

Roots of $2n^2+7n+3 = 0$ are $\dfrac{-7\pm\sqrt{49-24}}{4}$

i.e., $-\dfrac{1}{2}$ and -3 which are not natural numbers.

$\therefore$ n = 4 and hence sides are 4, 5, 6.

Example 2:

Consider the following statements concerning a ΔABC:

(i) The sides a, b, c and the area Δ are rational.

(ii) a, $\tan\dfrac{B}{2}$, $\tan\dfrac{C}{2}$ are rational.

(iii) a, sin A, sin B, sin C are rational.

Prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i)

Solution :

Let (i) be true, i.e., a, b, c and Δ be rational numbers.

Now, $\tan\dfrac{B}{2} = \dfrac{(s-c)(s-a)}{\Delta}$, $\tan\dfrac{C}{2} = \dfrac{(s-a)(s-b)}{\Delta}$ and $s = \dfrac{a+b+c}{2}$

Now, (i) $\Rightarrow$ a, b, c, Δ, s are rational.

So $\tan\frac{B}{2}$ and $\tan\frac{C}{2}$ are rational because sum, difference, product and quotient of nonzero rational numbers are rational.

Thus (i) $\Rightarrow$ (ii).

Let (ii) be true, i.e., a, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$ be rational

Now, $\sin B = \dfrac{2\tan\frac{B}{2}}{1+\tan^2\frac{B}{2}}$ = rational, because $\tan\frac{B}{2}$ is rational.

$\sin C = \dfrac{2\tan\frac{C}{2}}{1+\tan^2\frac{C}{2}}$ = rational, because $\tan\frac{C}{2}$ is rational.

Now, $\tan\frac{B}{2}.\tan\frac{C}{2} = \dfrac{(s-c)(s-a)}{\Delta}.\dfrac{(s-a)(s-b)}{\Delta} = \dfrac{(s-a)^2(s-b)(s-c)}{s(s-a)(s-b)(s-c)} = \dfrac{s-a}{s} = 1-\dfrac{a}{s}$.

$\therefore$ (ii) $\Rightarrow$ s is rational

$\Rightarrow$ b + c is rational, because a is rational.

But $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ $\therefore$ $\dfrac{a}{\sin A} = \dfrac{b+c}{\sin B+\sin C} = \dfrac{\text{rational}}{\text{rational}}$

$\therefore$ $\dfrac{a}{\sin A}$ is rational. But a is rational. So sinA is rational

Thus (ii) $\Rightarrow$ (iii)

Let (iii) be true, i.e., a, sin A, sin B, sin C be rational.

$\because$ $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$\therefore$ $b = \dfrac{a\sin B}{\sin A}$ = rational

and $c = \dfrac{a\sin C}{\sin A}$ = rational $\therefore$ $\Delta = \dfrac{1}{2}bc\sin A$ = rational.

Thus (iii) $\Rightarrow$ (i).

Example 3:

If in a triangle ABC, a = 6, b = 3 and cos (A – B) = $\frac{4}{5}$, find the area of the triangle.

Solution :

Here, $\cos(A-B) = \frac{4}{5}$, $\quad \therefore \quad \dfrac{1-\tan^2\frac{A-B}{2}}{1+\tan^2\frac{A-B}{2}} = \frac{4}{5}$

By componendo and dividendo, $\dfrac{2\tan^2\frac{A-B}{2}}{2} = \dfrac{5-4}{5+4}$ or $\tan^2\frac{A-B}{2} = \frac{1}{9}$

or $\quad \tan\frac{A-B}{2} = \frac{1}{3} \qquad (\because A > B).$

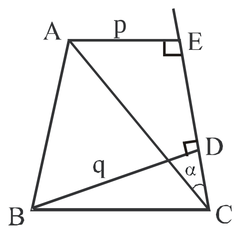
But $\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2} \quad \therefore \quad \frac{1}{3} = \frac{6-3}{6+3}\cot\frac{C}{2}$ or $\cot\frac{C}{2} = 1;$ $\therefore C = \frac{\pi}{2}.$

$\therefore \quad$ The area of the triangle $= \frac{1}{2}ab\sin C = \frac{1}{2}.6.3.\sin\frac{\pi}{2} = 9$ sq. units.

Example 4:

If p, q are perpendiculars from the angular points A and B of the Δ ABC drawn to any line through the vertex C, then prove that $a^2b^2\sin^2 C = a^2p^2 + b^2q^2 - 2abpq\cos C$.

Solution :



Let $\angle ACE = \alpha$. Clearly, from the figure, we get

$$\frac{p}{AC} = \sin\alpha, \frac{q}{BC} = \sin(\alpha + C)$$

$$\Rightarrow \quad \frac{p}{b} = \sin\alpha, \frac{q}{a} = \sin\alpha.\cos C + \cos\alpha.\sin C$$

$\therefore \quad \frac{q}{a} = \frac{p}{b}\cos C + \cos\alpha.\sin C$ or $\left(\frac{q}{a} - \frac{p}{b}\cos C\right)^2 = \cos^2\alpha.\sin^2 C = \left(1 - \frac{p^2}{b^2}\right)\left(1-\cos^2 C\right)$

or $\quad \frac{q^2}{a^2} + \frac{p^2}{b^2}\cos^2 C - \frac{2pq}{ab}\cos C = 1 - \frac{p^2}{b^2} - \left(1 - \frac{p^2}{b^2}\right)\cos^2 C$

or $\quad \frac{q^2}{a^2} + \frac{p^2}{b^2} - \frac{2pq}{ab}\cos C = \sin^2 C$ or $a^2p^2 + b^2q^2 - 2abpq\cos C = a^2b^2\sin^2 C$.

Example 5:

In a Δ ABC, prove that $\cos A.\cos C = \dfrac{2(c^2-a^2)}{3ca}$, where AD is the median through A and $AD \perp AC$.

Solution:

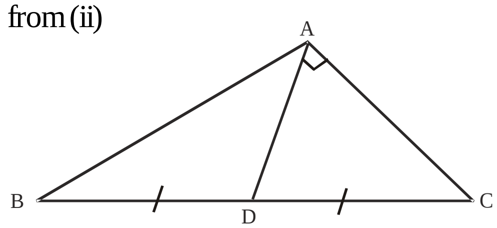
From the Δ ABC, $\cos A = \dfrac{b^2+c^2-a^2}{2bc} \qquad \ldots$ (i)

From the Δ CAD, $\cos C = \dfrac{AC}{CD} = \dfrac{b}{a/2} = \dfrac{2b}{a}$ . . . (ii)

From the Δ ABD, $\dfrac{BD}{\sin(A-90^0)} = \dfrac{AB}{\sin\angle ADB}$

or $\dfrac{a/2}{-\cos A} = \dfrac{c}{\sin(90^0+C)}$ or $\dfrac{a}{-2\cos A} = \dfrac{c}{\cos C}$

$\therefore$ $\cos A = \dfrac{a\cos C}{-2c} = \dfrac{a}{-2c}.\dfrac{2b}{a} = -\dfrac{b}{c}$, from (ii)



$\therefore$ from (i), $\dfrac{b^2+c^2-a^2}{2bc} = \dfrac{-b}{c}$

or $b^2 + c^2 - a^2 = -2b^2$

or $c^2 - a^2 = -3b^2$ . . . (iii)

Now, $\cos A.\cos C = \dfrac{b^2+c^2-a^2}{2bc}.\dfrac{2b}{a} = \dfrac{b^2+c^2-a^2}{ca}$

$= \dfrac{3b^2+3(c^2-a^2)}{3ca} = \dfrac{a^2-c^2+3(c^2-a^2)}{3ca}$, from (iii) $= \dfrac{2(c^2-a^2)}{3ca}$.

Example 6:

Find the sides and angles of the pedal triangle.

Solution:

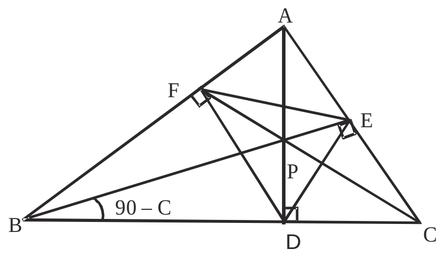
Since the angle PDC and PEC are right angles, the points P, E, C and D lie on a circle,

$\therefore$ $\angle PDE = \angle PCE = 90^0 - A$. Similarly P, D, B and F lie on a circle and therefore $\angle PDF = \angle PBF = 90^0 - A$, Hence $\angle FDE = 180^0 - 2A$

Similarly $\angle DEF = 180^0 - 2B$

$\angle EFD = 180^0 - 2C$

Also, from the triangle AEF we have



$$\frac{EF}{\sin A} = \frac{AE}{\sin\angle AFE} = \frac{AB\cos A}{\cos\angle PFE} = \frac{c\cos A}{\cos\angle PAE} = \frac{c\cos A}{\sin C}$$

$\therefore$ $EF = \dfrac{c}{\sin C}\sin A\cos A = a\cos A$

similarly DF = b cosB and DE = c cosC

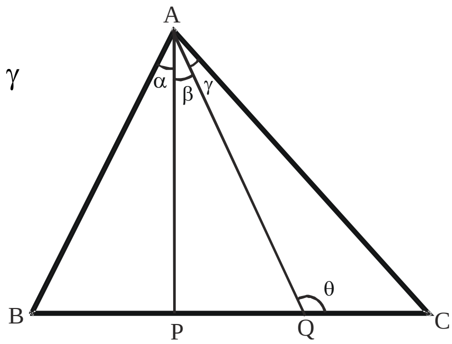
Example 7:

The base of a triangle is divided into three equal parts. If t_1, t_2, t_3 be the tangents of the angles subtended by these parts at the opposite vertex, prove that: $\left(\dfrac{1}{t_1}+\dfrac{1}{t_2}\right)\left(\dfrac{1}{t_2}+\dfrac{1}{t_3}\right) = 4\left(1+\dfrac{1}{t_2^2}\right)$

Solution:

Let the points P and Q divide the side BC in three equal parts such that BP = PQ = QC = x

Also let,



$\angle BAP = \alpha$, $\angle PAQ = \beta$, $\angle QAC = \gamma$

and $\angle AQC = \theta$

From question,

$\tan\alpha = t_1$, $\tan\beta = t_2$, $\tan\gamma = t_3$.

Applying,

m : n rule in triangle ABC, we get

$(2x + x)\cot\theta = 2x\cot(\alpha + \beta) - x\cot\gamma$ . . . (i)

from Δ APC, we get

$(x + x)\cot\theta = x\cot\beta - x\cot\gamma$ . . . (ii)

dividing (i) by (ii), we get

$$\frac{3}{2} = \frac{2\cot(\alpha+\beta) - \cot\gamma}{\cot\beta - \cot\gamma} \quad \text{or} \quad 3\cot\beta - \cot\gamma = \frac{4(\cot\alpha.\cot\beta - 1)}{\cot\beta + \cot\alpha}$$

or $\quad 3\cot^2\beta - \cot\beta\cot\gamma + 3\cot\alpha.\cot\beta - \cot\alpha.\cot\gamma = 4\cot\alpha.\cot\beta - 4$

or $\quad 4 + 4\cot^2\beta = \cot^2\beta + \cot\alpha.\cot\beta + \cot\beta.\cot\gamma + \cot\gamma.\cot\alpha$

or $\quad 4(1 + \cot^2\beta) = (\cot\beta + \cot\alpha)(\cot\beta + \cot\gamma)$

or $\quad 4\left(1 + \frac{1}{t_2^2}\right) = \left(\frac{1}{t_1} + \frac{1}{t_2}\right)\left(\frac{1}{t_2} + \frac{1}{t_3}\right)$

Hence the result.

***Example* 8:**

Perpendiculars are drawn from the angular points A, B and C of an acute angled Δ ABC on the oposite sides and produced to meet the circumscribing circle. If these produced parts be α, β and γ respectively, show that $\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 2(\tan A + \tan B + \tan C)$.

***Solution* :**

Let AD be perpendicular from A on BC. When AD is produced, it meets the circumscribing circle at E.

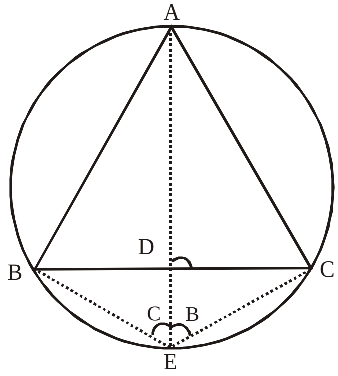
From question, DE = α.

Since, angle in the same segment are equal,

$\angle AEB = \angle ACB = C$ and $\angle AEC = \angle ABC = B$

From the right angled triangle BDE,

$$\tan C = \frac{BD}{DE} \qquad \ldots (i)$$



From the right angled triangle CDE,

$$\tan B = \frac{CD}{DE} \qquad \ldots (ii)$$

Adding (i) and (ii) we get, $\tan B + \tan C = \frac{a}{\alpha}$

Similarly $\tan C + \tan A = \frac{b}{\beta}$ and $\tan A + \tan B = \frac{C}{\gamma}$

$$\text{Hence, } \frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 2(\tan A + \tan B + \tan C)$$

Example 9:

If x, y, z are the distance of the vertices of the $\triangle$ABC respectively from the orthocentre then prove that $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{xyz}$.

Solution:

Let H be the orthocentre. Then

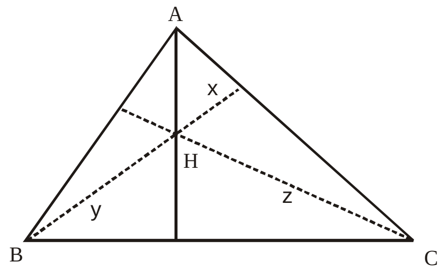
$$\angle BHC = 180^0 - \angle HBC - \angle HCB$$

$$= 180^0 - (90^0 - C) - (90^0 - B)$$

$$= B + C = \pi - A.$$

$$\therefore \quad ar(\triangle BHC) = \frac{1}{2} BH.\,CH \sin \angle BHC$$

$$= \frac{1}{2} yz \sin(\pi - A) = \frac{1}{2} yz\sin A.$$



Similarly, $ar(\triangle CHA) = \frac{1}{2} zx \sin B$

$$ar(\triangle AHB) = \frac{1}{2} xy \sin C$$

$$\therefore \quad ar(\triangle ABC) = \frac{1}{2} yz \sin A + \frac{1}{2} zx \sin B + \frac{1}{2} xy \sin C$$

$$= \frac{1}{2} xyz \left(\frac{\sin A}{x} + \frac{\sin B}{y} + \frac{\sin C}{z} \right)$$

$$= \frac{1}{2} xyz . \frac{1}{2R} \left(\frac{a}{x} + \frac{b}{y} + \frac{c}{z} \right) \qquad \ldots (i)$$

Also, we know that $R = \dfrac{abc}{4\Delta}$, i.e., $\Delta = \dfrac{abc}{4R}$

$\therefore$ (i) gives, $\dfrac{abc}{4R} = \dfrac{1}{4R} xyz\left(\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}\right)$ $\therefore$ $\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}=\dfrac{abc}{xyz}$.

Example 10:

Prove that in a Δ ABC, $R \geq 2r$.

Solution:

We have

$$r = 4R \sin A/2 \sin B/2 \sin C/2 \Rightarrow \frac{r}{4R} = \sin A/2 \sin B/2 \sin C/2$$

Also we know that $\sin A/2 \sin B/2 \sin C/2 \leq \dfrac{1}{8}$, $\Rightarrow \dfrac{r}{4R} \leq \dfrac{1}{8} \Rightarrow$ $R \geq 2r$.

Example 11:

Prove that in a triangle the sum of exradii exceeds the inradius by twice the diameter of the circumcircle.

Solution:

Let the exradii be r_1, r_2, r_3 inradius be r and circumradius be R.

Then we have to prove that $r_1 + r_2 + r_3 = r + 4R$.

Now, $r_1 + r_2 + r_3 - r = \dfrac{\Delta}{s-a}+\dfrac{\Delta}{s-b}+\dfrac{\Delta}{s-c}-\dfrac{\Delta}{s}$

$$= \Delta\left\{\left(\frac{1}{s-a}-\frac{1}{s}\right)+\left(\frac{1}{s-b}+\frac{1}{s-c}\right)\right\}$$

$$= \Delta\left\{\frac{a}{s(s-a)}+\frac{a}{(s-b)(s-c)}\right\}$$

$$= \Delta a\frac{s^2-s(b+c)+bc+s^2-as}{s(s-a)(s-b)(s-c)}$$

$$= \Delta a\frac{2s^2-s(a+b+c)+bc}{\Delta^2}$$

$$= \frac{a}{\Delta}.\left(2s^2-2s^2+bc\right)=\frac{abc}{\Delta}=4R \quad \left\{\because R=\frac{abc}{4\Delta}\right\}$$

$\therefore$ $r_1 + r_2 + r_3 = r + 4R$.

Example 12:

If a, b, c are in A.P., prove that cos A cot A/2, cosB. cot B/2, cosCcot C/2 are in A.P.

Solution:

a, b, c are in A.P.

$\Rightarrow$ cotA/2, cot B/2, cotC/2 are in A.P.

Now, cosA cotA/2, cosB cotB/2, cosC cotC/2 are

$(1 - 2\sin^2 A/2)\cot A/2, (1 - 2\sin^2 B/2)\cot B/2, (1 - 2\sin^2 C/2).\cot C/2$

Now, cot A/2 – sinA, cotB/2 – sin B, cotC/2 – sin C are in A.P. as cotA/2, cotB/2, cotC/2 are in A.P. and sinA, sin B, sin C are in A.P.

So, cos A cot A/2, cosB. cot B/2, cosCcot C/2 are in A.P.

Example 13:

If r and R are radii of the incircle and circumcircle of a Δ ABC, prove that

$8rR\{\cos^2 A/2 + \cos^2 B/2 + \cos^2 C/2\} = 2bc + 2ca + 2ab - a^2 - b^2 - c^2$.

Solution:

$$\text{L.H.S.} = 8\left(\frac{\Delta}{s}\right).\left(\frac{abc}{4\Delta}\right)\left\{\sum \cos^2 A/2\right\} = \frac{abc}{s}\sum\left(2\cos^2 A/2\right)$$

$$= \frac{abc}{s}\sum(1+\cos A)$$

$$= \frac{abc}{s}\sum\left(1+\frac{b^2+c^2-a^2}{2bc}\right)$$

$$= \frac{abc}{s}\sum\left(\frac{2bc+b^2+c^2-a^2}{2bc}\right)$$

$$= \frac{abc}{s}\sum\left(\frac{(b+c)^2-a^2}{2bc}\right)$$

$$= \frac{abc}{s}\sum\left(\frac{(a+b+c)(b+c-a)}{2bc}\right), \qquad \text{where } a+b+c = 2s$$

$$= \frac{abc2s}{s}\sum\left(\frac{(b+c-a)}{2bc}\right) = \sum a(b+c-a) = \sum\left(ab+bc-a^2\right)$$

$$= 2bc+2ca+2ab-a^2-b^2-c^2$$

$\therefore$ $8rR\{\cos^2 A/2 + \cos^2 B/2 + \cos^2 C/2\} = 2bc + 2ab + 2ca - a^2 - b^2 - c^2$.

Example 14:

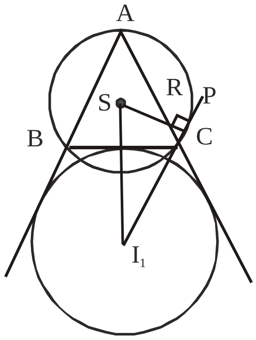
If 't_1', 't_2' and 't_3' are the lengths of the tangents drawn from centre of ex-circle to the circum circle of the Δ ABC, then prove that $\frac{1}{t_1^2}+\frac{1}{t_2^2}+\frac{1}{t_3^2} = \frac{abc}{a+b+c}$

Solution:

Let S and I_1 be respectively the centres of the circumcircle and the excircle touching BC. It can be shown that

$SI_1 = \sqrt{R^2 + 2Rr_1}$ In Δ SI_1P, $SI_1^2 = R^2 + t_1^2$

$R^2 + 2Rr_1 = R^2 + t_1^2$, $\frac{1}{t_1^2} = \frac{1}{2Rr_1}$



Similarly $\frac{1}{t_2^2} = \frac{1}{2Rr_2}, \frac{1}{t_3^2} = \frac{1}{2Rr_3}$

$$\frac{1}{t_1^2} + \frac{1}{t_2^2} + \frac{1}{t_3^2} = \frac{1}{2R}\left[\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right]$$

$$= \frac{1}{2R}\left[\frac{s-a}{\Delta} + \frac{s-b}{\Delta} + \frac{s-c}{\Delta}\right] = \frac{1}{2R}\frac{s}{\Delta} = \frac{s}{2R\Delta}$$

$$= \frac{a+b+c}{abc}$$ proved

Example 15:

If a, b and A are given in a triangle and c_1, c_2 are the possible values of the third side, prove that

$c_1^2 + c_2^2 - 2c_1c_2 \cos A = 4a^2 \cos^2 A$

Solution:

We have $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

$\Rightarrow$ $c^2 - 2bc \cos A + b^2 - a^2 = 0$, which is quadratic in 'c'

SOLVED OBJECTIVE EXAMPLES

***Example* 1:**

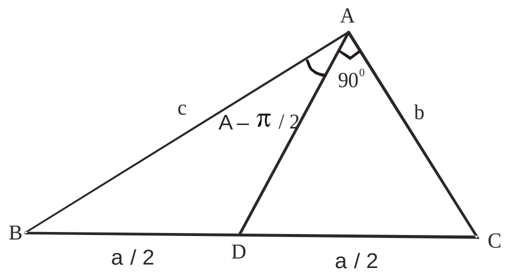
If D is the mid point of the side BC of a triangle ABC and AD is perpendicular to AC, then

(A) $3b^2 = a^2 - c^2$ (B) $3a^2 = b^2 - 3c^2$

(C) $b^2 = a^2 - c^2$ (D) $a^2 + b^2 = 5c^2$

Solution:

From the right angled ΔCAD, we have



$$\cos C = \frac{b}{a/2} \Rightarrow \frac{2b}{a} = \frac{a^2+b^2-c^2}{2ab}$$

$a^2 + b^2 - c^2 = 4b^2 \quad \Rightarrow \quad a^2 - c^2 = 3b^2.$

***Example* 2:**

There exists a triangle ABC satisfying

(A) $\tan A + \tan B + \tan C = 0$ (B) $\frac{\sin A}{2} = \frac{\sin B}{3} = \frac{\sin C}{7}$

(C) $(a + b)^2 = c^2 + ab$ (D) none of these

Solution:

(A) In a triangle ABC, we know that $\tan A + \tan B + \tan C = \tan A \tan B \tan C$. Since none of $\tan A$, $\tan B$, $\tan C$ can be zero, (A) is not possible

If $(\sin A)/2 = (\sin B)/3 = (\sin C)/7$, then by the laws of sines $\frac{a}{2} = \frac{b}{3} = \frac{c}{7}$

which is not possible, as the sum of two sides of a triangle is greater than the third side

If $(a + b)^2 = c^2 + ab$, then $\frac{a^2+b^2-c^2}{2ab} = \frac{1}{2} = \cos C = \frac{\pi}{3}$, which is possible

Hence (C) is the correct answer.

***Example* 3:**

If the tangents of the angles A and B of a triangle ABC satisfy the equation $abx^2 - c^2x + ab = 0$, then

(A) $\tan A = a/b$ (B) $\tan B = b/a$

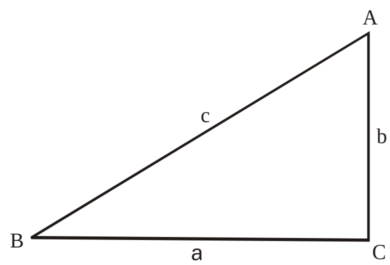
(C) $\cos C = 0$ (D) $\sin^2 A + \sin^2 B + \sin^2 C = 2$

Solution:

From the given equation, we get

$\tan A + \tan B = c^2 / ab$ and $\tan A \tan B = 1$.

Since $\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$



We get $A + B = \frac{\pi}{2}$ and hence $C = \frac{\pi}{2}$.

Therefore, triangle ABC is right angled at C. Hence, tan A = a/b, tan B = b/a, cos C = 0, sin A = a/c, sin B = b/c and sin C = 1, so that

$\sin^2 A + \sin^2 B + \sin^2 C = \frac{a^2}{c^2} + \frac{b^2}{c^2} + 1 = \frac{a^2+b^2}{c^2} + 1 = 1 + 1 = 2$ [$\because a^2 + b^2 = c^2$]

Hence, all options are correct.

Example 4:

If in a triangle ABC sin A, sin B and sin C are in A.P., then the altitudes are in

(A) A.P.
(B) H.P.
(C) G.P.
(D) none of these

Solution:

If p_1, p_2, p_3, are altitude from A, B, C respectively,

then $\Delta = \frac{1}{2} ap_1 = \frac{1}{2} bp_2 = \frac{1}{2} cp_3 \Rightarrow p_1 = \frac{2\Delta}{a}, p_2 = \frac{2\Delta}{b}, p_3 = \frac{2\Delta}{c}$

By the law of sines

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$ (say)

$\therefore \quad p_1 = \frac{2\Delta}{k \sin A}, p_2 = \frac{2\Delta}{k \sin B}, p_3 = \frac{2\Delta}{k \sin C}$

Now, sin A, sin B, sin C are in A.P. $\Rightarrow$ p_1, p_2, p_3 are in H.P.

Example 5:

In a triangle ABC, medians AD and CE are drawn. If AD = 5, $\angle DAC = \pi/8$ and $\angle ACE = \pi/4$, then the area of the triangle ABC is equal to

(A) $\frac{25}{9}$
(B) $\frac{25}{3}$
(C) $\frac{25}{18}$
(D) $\frac{10}{3}$

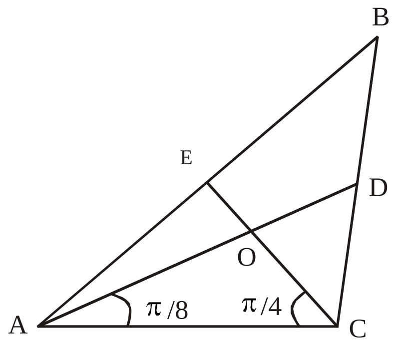
Solution:

Let O be the point of intersection of the medians of triangle ABC. Then the area of Δ ABC is three times that of Δ AOC. Now, in Δ AOC, AO = $\frac{2}{3}$AD = $\frac{10}{3}$. Therefore, applying the sine rule to Δ AOC, we get

$$\frac{OC}{\sin(\pi/8)} = \frac{AO}{\sin(\pi/4)} \Rightarrow OC = \frac{10}{3} . \frac{\sin(\pi/8)}{\sin(\pi/4)}$$

area of Δ AOC = $\frac{1}{2}$. AO.OC. sin $\angle$AOC

$$= \frac{1}{2}.\frac{10}{3}.\frac{10}{3}.\frac{\sin(\pi/8)}{\sin(\pi/4)}.\sin\left(\frac{\pi}{2}+\frac{\pi}{8}\right)$$



$$= \quad \frac{50}{9}.\frac{\sin(\pi/8)\cos(\pi/8)}{\sin(\pi/4)} = \frac{50}{18} = \frac{25}{9}$$

$\therefore$ area of Δ ABC = 3. $\frac{25}{9} = \frac{25}{3}$

Example 6:

In a triangle ABC, if tan (A/2) = 5/6 and tan (B/2) = 20/37, the sides a, b and c are in

(A) A.P. (B) G.P.

(C) H.P (D) none of these

Solution:

We have $\tan\frac{C}{2} = \tan\left(90^0 - \frac{A+B}{2}\right) = \cot\frac{A+B}{2} = \frac{\cot(A/2)\cot(B/2)-1}{\cot(A/2)+\cot(B/2)}$

$$= \frac{\frac{6}{5}.\frac{37}{20}-1}{\frac{6}{5}+\frac{37}{20}} = \frac{222-100}{120+185} = \frac{122}{305} = \frac{2}{5}$$

Also $\tan\frac{A}{2}\tan\frac{C}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

$\Rightarrow \frac{5}{6}.\frac{2}{5} = \frac{s-b}{s} \quad \Rightarrow \quad 3(s-b) = s \quad \Rightarrow 2s = 3b$

$\Rightarrow$ a + b + c = 3b $\Rightarrow$ a + c = 2b

Which shows that a, b and c are in A.P.

Example 7:

If in a triangle ABC, a = 5, b = 4 and cos (A – B) = 31/32, then the third side c is equal to

(A) 6 (B) 8

(C) 4 (D) none of these

Solution:

$$\cos(A-B) = \frac{1-\tan^2\frac{A-B}{2}}{1+\tan^2\frac{A-B}{2}} \quad \Rightarrow \frac{31}{32} = \frac{1-\tan^2\frac{A-B}{2}}{1+\tan^2\frac{A-B}{2}}$$

$$\Rightarrow \quad 63\tan^2\frac{A-B}{2} = 1 \quad \Rightarrow \quad \tan\frac{A-B}{2} = \frac{1}{\sqrt{63}}$$

Now $\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2} \Rightarrow \frac{1}{\sqrt{63}}=\frac{5-4}{5+4}\cot\frac{C}{2}$

$\Rightarrow \quad \tan\frac{C}{2} = \frac{\sqrt{63}}{9}$

Also, $\cos C = \frac{1-\tan^2(C/2)}{1+\tan^2(C/2)}=\frac{1-63/81}{1+63/81}=\frac{18}{144}=\frac{1}{8}$

$c^2 = a^2 + b^2 - 2ab\cos C = 25 + 16 - 2.5.4.(1/8) = 36 \Rightarrow c = 6$

Hence (A) is the correct answer.

Example 8:

In a triangle ABC, if $r_1 = 2r_2 = 3r_3$, then a : b is equal to

(A) $\frac{5}{4}$ (B) $\frac{4}{5}$

(C) $\frac{7}{4}$ (D) $\frac{4}{7}$

Solution:

From the given relation, we have

$s\tan\frac{A}{2} = 2s\tan\frac{B}{2} = 3s\tan\frac{C}{2}$

$\frac{\tan(A/2)}{6}=\frac{\tan(B/2)}{3}=\frac{\tan(c/2)}{2}=k$ (say)

Also, since $A/2 + B/2 + C/2 = 90^0$, we get $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$

$\Rightarrow 6k.\ 3k + 3k.\ 2k + 2k.\ 6k = 1 \Rightarrow 36k^2 = 1 \Rightarrow k = 1/6$

$\Rightarrow \quad \sin A = \frac{2\tan(A/2)}{1+\tan^2(A/2)}=\frac{12k}{1+36k^2}=1$

and $\sin B = \frac{2\tan(B/2)}{1+\tan^2(B/2)}=\frac{6k}{1+9k^2}=\frac{4}{5}$

Hence, by the law of sines, $\sin A/a = \sin B/b$, we have

$\Rightarrow \quad \frac{a}{b}=\frac{\sin A}{\sin B}=\frac{5}{4} \quad \Rightarrow \quad a : b = 5 : 4$

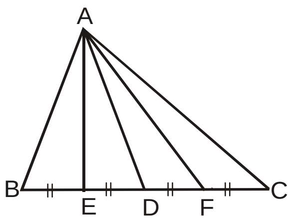
Example 9:

Let AD be a median of the ΔABC. If AE and AF are medians of the triangles ABD and ADC respectively and $AD = m_1$, $AE = m_2$, $AF = m_3$, then $\frac{a^2}{8}$ is equal to

(A) $m_2^2 + m_3^2 - 2m_1^2$ (B) $m_1^2 + m_2^2 - 2m_3^2$

(C) $m_2^2 + m_3^2 - 2m_1^2$ (D) none of these

Solution:



In Δ ABC, $AD^2 = m_1^2 = \dfrac{c^2+b^2}{2} - \dfrac{a^2}{4}$

In Δ ABD, $AE^2 = m_2^2 = \dfrac{AD^2+c^2}{2} - \dfrac{\left(\frac{a}{2}\right)^2}{4}$

$$AF^2 = m_3^2 = \dfrac{AD^2+b^2}{2} - \dfrac{\left(\frac{a}{2}\right)^2}{4}$$

$\therefore\ m_2^2 + m_3^2 = AD^2 + \dfrac{b^2+c^2}{2} - \dfrac{a^2}{8} = m_1^2 + m_1^2 + \dfrac{a^2}{4} - \dfrac{a^2}{8} = 2m_1^2 + \dfrac{a^2}{8}$

$\therefore\ m_2^2 + m_3^2 - 2m_1^2 = \dfrac{a^2}{8}$

***Example* 10:**

If I is the incentre of a triangle ABC, then the ratio IA : IB : IC is equal to

(A) $\text{cosec}\dfrac{A}{2} : \text{cosec}\dfrac{B}{2} : \text{cosec}\dfrac{C}{2}$ (B) $\sin\dfrac{A}{2} : \sin\dfrac{B}{2} : \sin\dfrac{C}{2}$

(C) $\sec\dfrac{A}{2} : \sec\dfrac{B}{2} : \sec\dfrac{C}{2}$ (D) none of these

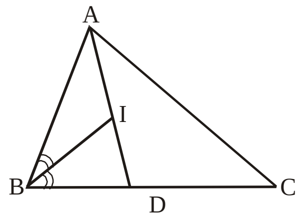
Solution:

Here BD : DC = c : b

But BD + DC = a;

$\therefore\ BD = \dfrac{c}{b+c}.a$

In Δ ABD, $\dfrac{BD}{\sin\frac{A}{2}} = \dfrac{AD}{\sin B}$



$\therefore\ AD = \dfrac{ca}{b+c}.\dfrac{\sin B}{\sin\frac{A}{2}} = \dfrac{2\Delta}{b+c}\text{cosec}\dfrac{A}{2}$

Also, $\frac{AI}{ID} = \frac{AB}{BD} = \frac{c}{ca/(b+c)} = \frac{b+c}{a}$

$\Rightarrow \quad AI = \frac{b+c}{a+b+c}.AD = \frac{\Delta}{s}\operatorname{cosec}\frac{A}{2}$ Similarly $BI = \frac{\Delta}{s}\operatorname{cosec}\frac{B}{2}$, $CI = \frac{\Delta}{s}\operatorname{cosec}\frac{C}{2}$

$\therefore \quad IA : IB : IC = \operatorname{cosec}\frac{A}{2} : \operatorname{cosec}\frac{B}{2} : \operatorname{cosec}\frac{C}{2}$

***Example* 11:**

In a ΔABC, the value of $\frac{a\cos A + b\cos B + c\cos C}{a+b+c}$ is equal to

(A) $\frac{R}{r}$ (B) $\frac{R}{2r}$

(C) $\frac{r}{R}$ (D) $\frac{2r}{R}$

Solution:

$$\frac{a\cos A + b\cos B + c\cos C}{a+b+c} = \frac{2R\sin A\cos A + 2R\sin B\cos B + 2R\sin C\cos C}{2s}$$

$$= \frac{R}{2s}.(\sin 2A + \sin 2B + \sin 2C) = \frac{R}{2s}.4\sin A \sin B \sin C = \frac{4R}{2s}.\frac{abc}{8R^3} = \frac{abc}{4sR^2}$$

But $R = \frac{abc}{4\Delta}$, $r = \frac{\Delta}{s}$. So, the value $= \frac{4\Delta R}{4.\frac{\Delta}{r}.R^2} = \frac{r}{R}$

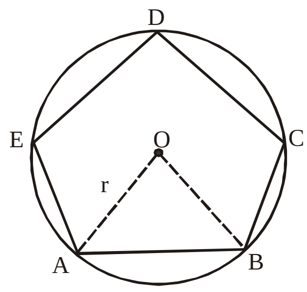
***Example* 12:**

The area of a circle is A_1 and the area of a regular pentagon inscribed in the circle is A_2. Then $A_1 : A_2$ is

(A) $\frac{\pi}{5}\cos\frac{\pi}{10}$ (B) $\frac{2\pi}{5}\sec\frac{\pi}{10}$

(C) $\frac{2\pi}{5}\operatorname{cosec}\frac{\pi}{10}$ (D) none of these

Solution:



In the ΔOAB, $OA = OB = r$ and $\angle AOB = \frac{360^0}{5} = 72^0$

$\therefore$ ar (ΔAOB) $= \frac{1}{2}$. r . r. sin $72^0 = \frac{1}{2} r^2 \cos 18^0$

$\therefore \quad A_1 : A_2 = \frac{2\pi r^2}{5r^2\cos 18^0} = \frac{2\pi}{5}\sec\frac{\pi}{10}$

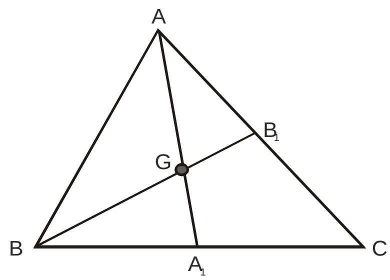
Example 13:

In a triangle ABC a = 5, b = 4 and c = 3. 'G' is the centroid of the triangle. Circumradius of triangle GAB is equal to

(A) $2\sqrt{13}$ (B) $\frac{5}{12}\sqrt{13}$

(C) $\frac{5}{3}\sqrt{13}$ (D) $\frac{3}{2}\sqrt{13}$

Solution:



$AG = \frac{2}{3}AA_1$, $BG = \frac{2}{3}BB_1$

$\Rightarrow AG = \frac{1}{3}\sqrt{2b^2 + 2c^2 - a^2}$

and $BG = \frac{1}{3}\sqrt{2a^2 + 2c^2 - b^2}$

$\Rightarrow AG = \frac{1}{3}a, BG = \frac{1}{3}\sqrt{b^2 + 4c^2}$ as $a^2 = b^2 + c^2$

$\Rightarrow AG = \frac{5}{3}$, $BG = \frac{1}{3}\sqrt{16+36} = \frac{2}{3}\sqrt{13}$

Also, AB = c = 3 and $\Delta_{GAB} = \frac{1}{3}\Delta_{ABC} = 2$

If 'R_1' be the circumradius of triangle GAB then

$R_1 = \frac{(AG)(BG)(AB)}{4\Delta_{GAB}} = \frac{5}{3}.\frac{2}{3}\sqrt{13}.3.\frac{1}{4.2}$

$= \frac{5\sqrt{13}}{12}$ units.

Example 14:

A variable triangle ABC is circumscribed about a fixed circle of unit radius. Side BC always touches the circle at D and has fixed direction. If B and C vary in such a way that (BD). (CD) = 2 then locus of vertex A will be a straight line

(A) parallel to side BC (B) right angle to side BC

(C) making an angle $\pi/6$ with BC (D) making an angle $\sin^{-1}(2/3)$ with BC

Solution:

BD = (s – b), CD = (s – c) $\Rightarrow$ (s – b)(s – c) = 2

$\Rightarrow$ s(s – a) (s – b) (s – c) = 2 s(s – a)

$\Rightarrow \Delta^2 = 2\,s(s-a) \Rightarrow \frac{\Delta^2}{s^2} = \frac{2(s-a)}{s} = 1$ (radius of incircle of triangle ABC)

$\Rightarrow \frac{a}{s} = \text{constant.}$

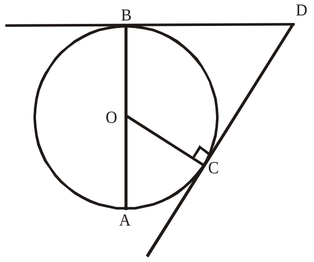
Now $\Delta = \frac{1}{2} aH_a$, where '$H_a$' is the distance of 'A' from BC.

$\Rightarrow \frac{\Delta}{s} = \frac{1}{2}\frac{aH_a}{s} = 1 \Rightarrow H_a = \frac{2s}{a} = \text{constant}$

$\Rightarrow$ Locus of 'A' will be a straight line parallel to side BC.

Example 15:

In the adjacent figure AB is the diameter of circle, centered at 'O'. If $\angle COA = 60^0$. AB = 2r, AC = d and CD = ℓ, then



(A) $\sqrt{3}\ell = r = d$

(B) $\ell = r\sqrt{2} = d\sqrt{2}$

(C) $\ell = r\sqrt{3} = d\sqrt{3}$

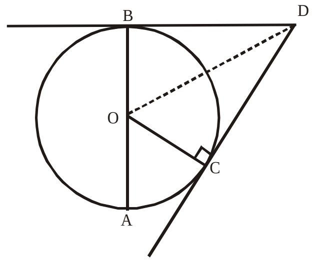
(D) $\sqrt{2}\ell = r = d$

Solution:

AC = d, OA = OB = r , CD = BD = ℓ, $\angle COA = \frac{\pi}{3}$

$\therefore AC^2 = OA^2 + OC^2 - 2AOOC.\cos\frac{\pi}{3}$

$\Rightarrow d^2 = 2r^2 - 2r^2 . \frac{1}{2} = r^2$



Also, $\angle BOD = \angle COD = \frac{2\pi}{3.2} = \frac{\pi}{3}$

$\Rightarrow \tan\frac{\pi}{3} = \frac{BD}{OB} = \frac{\ell}{r} \Rightarrow \ell = r\sqrt{3} = d\sqrt{3}$

Hence the correct answer is (C)

$$\therefore \quad \left.\begin{aligned} c_1 + c_2 &= 2b\cos A \\ \text{and} \quad c_1c_2 &= b^2 - a^2 \end{aligned}\right\} \quad \ldots (i)$$

$\therefore \quad c_1^2 + c_2^2 - 2c_1c_2 \cos 2A$

$\Rightarrow \quad (c_1 + c_2)^2 - 2c_1c_2 - 2c_1c_2 \cos 2A$ [using (i)]

$\Rightarrow \quad (c_1 + c_2)^2 - 2c_1c_2 (1 + \cos 2A)$

$\Rightarrow \quad 4b^2 \cos^2A - 2(b^2 - a^2). 2\cos^2A = 4a^2 \cos^2A$

$\therefore \quad c_1^2 + c_2^2 - 2c_1c_2 \cos A = 4a^2 \cos^2 A$