SOLVED SUBJECTIVE EXAMPLES

Example 1 :

The area of an expanding rectangle is increasing at the rate of 48 cm^2/sec. The length of the rectangle is always equal to the square of the breadth. At what rate the length is increasing at the instant when the breadth is 4.5 cm?

Solution:

Let l and b be respectively the length and breadth of the rectangle at time t.

$\therefore \quad l = b^2$

Let A be the area of the rectangle at time t

$\therefore \quad A = l \times b = b^2 \times b = b^3$

The area is increasing at the rate of 48 cm^2 /sec.

$$\Rightarrow \quad \frac{dA}{dt} = 48 \Rightarrow \frac{d}{dt}(b^3) = 48 \Rightarrow 3b^2\frac{db}{dt} = 48 \Rightarrow \frac{db}{dt} = \frac{16}{b^2}$$

Rate of change of length w.r.t t = $\frac{dl}{dt} = \frac{d}{dt}(b^2) = 2b\frac{db}{dt} = 2b \times \frac{16}{b^2} = \frac{32}{b}$ (a + ve quantity)

$\therefore$ When b = 4.5, the rate of increase ($\because$ dl/dt is +ve) of length = $\frac{32}{4.5}$ = 7.11 cm./sec.

Example 2 :

Find the points on the curve $ay^2 = x^3$ where normal to the curve makes equal intercepts with the axes.

Solution:

Let the point at which normal is drawn be (x_1, y_1). Then it must satisfy $ay^2 = x^3$,

i.e., $ay_1^2 = x_1^3$ or $y_1 = \pm\sqrt{\frac{x_1^3}{a}}$

Now, differentiating both sides of the given curve with respect to x we get,

$$2ay\frac{dy}{dx} = 3x^2$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \frac{3x_1^2}{2ay_1} = \frac{3x_1^2}{\pm 2a\sqrt{\frac{x_1^3}{a}}} = \pm\frac{3}{2}\sqrt{\frac{x_1}{a}} \qquad ..(1)$$

Thus, slope of the normal $\Rightarrow -\left(\frac{dx}{dy}\right)_{(x_1,y_1)} = \pm\frac{2}{3}\sqrt{\frac{a}{x_1}}$.

We know that the slope of the line making equal intercept with the axes = $\pm$ 1.

$\Rightarrow \quad \pm\frac{2}{3}\sqrt{\frac{a}{x_1}} = \pm 1 \quad \Rightarrow \quad x_1 = \frac{4a}{9}$

Hence, the required points are $\left(\frac{4a}{9}, \frac{8a}{27}\right)$ and $\left(\frac{4a}{9}, \frac{-8a}{27}\right)$

Example 3 :

Prove that all the normals to the curve x = a cost + at sint and y = a sint – at cost are at a distance a from the origin $(a \in R^+)$.

Solution:

x = a cost + at sint

$$\Rightarrow \quad \frac{dx}{dt} = -a\,\text{sint} + at\cos t + a\sin t = at\cos t$$

$$y = a\sin t - at\cos t \Rightarrow \frac{dy}{dt} = a\,\text{cost} + at\sin t - a\text{cost} = at\sin t$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{at\ \sin\ t}{at\ \cos\ t} = \text{tant}$$

Hence the equation of the normal at any point 't' on the curve is

$$(y - (a\,\text{sint} - at\,\text{cost})) = -\frac{1}{\tan t}(x - (a\,\text{cost} + at\,\text{sint}))$$

$\Rightarrow \quad y\text{sint} - a\sin^2 t + at\,\text{sint cost} = -x\text{cost} + a\cos^2 t + at\,\text{sint cost}$

or $\quad x\,\text{cost} + y\text{sint} = a$

Distance of the normal from (0, 0)= $\frac{|a|}{\sqrt{\sin^2 t + \cos^2 t}} = a$, as $a \in R^+$

Example 4 :

Check whether the following functions satisfy the conditions of Rolle's theorem

(a) $\quad f(x) = x^2 - \sqrt[3]{|x|}, x \in [-1,1]$

(b) $\quad f(x) = \sec x,\ x \in \left[\frac{-\pi}{4}, \frac{\pi}{3}\right]$

Solution:

(a) We have

$$f(x) = \begin{cases} x^2 + \sqrt[3]{x}\ , & -1 \le x < 0 \\ x^2 - \sqrt[3]{x}\ , & 0 < x \le 1 \end{cases}$$

which is continuous in [–1, 1]. Also, we have

$f(-1) = 0 = f(1)$

Differentiating f(x) w.r.t. x, we have

$$f'(x) = 2x + \frac{1}{3x^{2/3}}, \ -1 \le x < 0, \ = 2x - \frac{1}{3x^{2/3}}, \ 0 < x \le 1$$

which is not differentiable at x = 0

Hence, Rolle's theorem is not applicable to the given function.

(b) We have

$$f(x) = \sec x, x \in \left[\frac{-\pi}{4}, \frac{\pi}{3}\right]$$

Differentiating w.r.t. x, we have

$$f'(x) = \sec x \tan x = \frac{\sin x}{\cos^2 x}$$

Thus, f(x) is continuous in $\left[\frac{-\pi}{4}, \frac{\pi}{3}\right]$ and differentiable in $\left(\frac{-\pi}{4}, \frac{\pi}{3}\right)$ but

$f\left(\frac{-\pi}{4}\right) = \sqrt{2}$ is not equal to $f\left(\frac{\pi}{3}\right) = 2$

Hence, Rolle's theorem is not applicable to the given function.

***Example* 5 :**

Show that the radius of the right circular cylinder of greatest curved surface which can be inscribed in a given cone is half that of the cone.

***Solution*:**

Let 'b' be the height of the cone and α be its semiverticle angle.

LD = x = radius of the inscribed cylinder and LM = h be its height

LM = OM – OL = b – x cot α

Now, $S = 2\pi rh$ = curved surface

$S = 2\pi x (b - x \cot \alpha)$

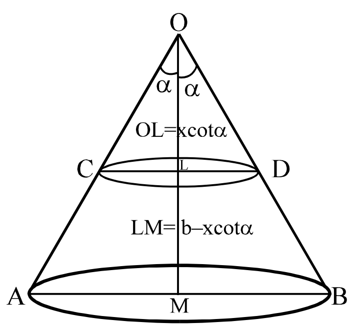
or $S = 2\pi (bx - x^2 \cot \alpha)$

$\therefore$ $dS/dx = 2\pi (b - 2x \cot \alpha) = 0$

$\therefore$ $x = (b/2) \tan \alpha$

or $x = \frac{1}{2}$ (b tan α) = $\frac{1}{2}$ (r_1)

or Radius of cylinder $\left(\frac{1}{2}\right)$.(radius of cone)



Example 6 :

Prove that $(\beta - \alpha)\sec^2\alpha < \tan\beta - \tan\alpha < (\beta - \alpha)\sec^2\beta$, where $0 < \alpha < \beta < \frac{\pi}{2}$

Solution:

Let $f(x) = \tan x$

we know that tanx is continuous and differentiable function in $(0, \pi/2)$, so according to LMVT,

there exists a point 'γ' in (α, β) where

$$f'(\gamma) = \frac{f(\beta) - f(\alpha)}{\beta - \alpha} = \frac{\tan\beta - \tan\alpha}{\beta - \alpha}$$

Also, $f'(x) = \sec^2 x$, $f''(x) = 2\sec^2 x \tan x > 0$

So, $f'(x)$ is increasing in $(0, \pi/2)$

$\Rightarrow \quad f'(\alpha) < f'(\gamma) < f'(\beta)$

$\Rightarrow \quad \sec^2\alpha < \frac{\tan(\beta) - \tan(\alpha)}{(\beta - \alpha)} < \sec^2\beta$

which proves the required result.

Example 7 :

Prove that the following functions are increasing in the given intervals,

$y = e^x + \sin x$, $x \in R^+$

Solution:

(i) $\quad f(x) = e^x + \sin x$, $x \in R^+$

$\Rightarrow \quad f'(x) = e^x + \cos x$

Clearly $f'(x) > 0 \ \forall x \in R^+$ (as $e^x > 1$, $x \in R^+$ and $-1 \le \cos x \le 1$, $x \in R^+$)

Hence f(x) is increasing.

Example 8 :

Let S be the non-empty set containing all a for which $f(x) = \frac{4a - 7}{3}x^3 + (a - 3)x^2 + x + 5$ is monotonic for all $x \in R$. Find S.

Solution:

Let $f(x) = \frac{4a - 7}{3}x^3 + (a - 3)x^2 + x + 5$

$\Rightarrow f'(x) = (4a - 7)x^2 + 2(a - 3)x + 1$

For f(x) to be monotonic, $f'(x) \geq 0$ or $f'(x) \leq 0$

$\Rightarrow D \leq 0 \Rightarrow (a-3)^2 - (4a-7) \leq 0 \Rightarrow a^2 - 10a + 16 \leq 0$

$\Rightarrow (a-2)(a-8) \leq 0 \Rightarrow a \in [2, 8]$

Also $4a - 7 \neq 0 \Rightarrow a \neq 7/4$

So $a \in [2, 8] - \left\{\frac{7}{4}\right\} \Rightarrow S = [2, 8] - \left\{\frac{7}{4}\right\}$.

Example **9 :**

(i) Using Calculus, find the order relation between x and $\tan^{-1}x$.

(ii) Show that $\ln(1 + x) < x$ for all $x > 0$

Solution**:**

(i) Let $f(x) = x - \tan^{-1}x \Rightarrow f'(x) = 1 - \frac{1}{1+x^2} = \frac{x^2}{1+x^2} \geq 0 \; \forall x \in R$

Thus f(x) is a increasing function.

Now, $f(0) = 0$

$\Rightarrow \quad f(x) < 0, \; \forall x \in (-\infty, 0)$ and $f(x) \geq 0, \; x \in [0, \infty)$

$\Rightarrow \quad x < \tan^{-1}x, \; x \in (-\infty, 0)$ and $x \geq \tan^{-1}x, \; x \in [0, \infty)$

(ii) Let us assume $f(x) = \ln(1 + x) - x$

$$\Rightarrow \quad f'(x) = \frac{1}{1+x} - 1 = \frac{-x}{1+x}$$

clearly, $f'(x) < 0 \; \forall x \in (0, \infty)$

Hence for $x > 0$ f(x) is decreasing

Moreover $f(0) = 0$, hence further

$f(x) < 0 \Rightarrow \ln(1 + x) - x < 0$

$\Rightarrow \quad \ln(1 + x) < x$ for all $x > 0$

Example **10 :**

Find the values of a, if the equation $x - \sin x = a$ has a unique root in $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

Solution**:**

Consider the function

$$f(x) = x - \sin x - a, \; x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$

Then $f'(x) = 1 - \cos x = 2\sin^2\left(\frac{x}{2}\right) > 0 \; \forall x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$\Rightarrow$ f(x) strictly increases in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

Also, we have

$$f\left(\frac{-\pi}{2}\right) = \frac{-\pi}{2} + 1 - a$$

and $$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - 1 - a$$

The curve y = f(x) will cuts the x-axis exactly once, if $f\left(\frac{-\pi}{2}\right)$ is negative or zero and $f\left(\frac{\pi}{2}\right)$ is positive or zero.

i.e., $\frac{-\pi}{2} + 1 - a \le 0$ and $\frac{\pi}{2} - 1 - a > 0$

i.e., $a \ge \frac{-\pi}{2} + 1$ and $a < \frac{\pi}{2} - 1$

Hence, we have , $a \in \left[1 - \frac{\pi}{2}, \frac{\pi}{2} - 1\right]$

***Example* 11 :**

Find the shortest distance between the curves $9x^2 + 9y^2 - 30y + 16 = 0$ and $y^2 = x^3$.

***Solution*:**

$9x^2 + 9y^2 - 30y + 16 = 0$ can be rewritten as $x^2 + \left(y - \frac{5}{3}\right)^2 = 1$

Any point on the curve $y^2 = x^3$ can be taken as (t^2, t^3).

Let L be the distance between the centre of the given circle and the point (t^2, t^3), then

$$K = L^2 = t^4 + (t^3 - 5/3)^2$$

Now, we calculate the minimum value of L. Required distance = L – radius of given circle.

Now, $\frac{dK}{dt} = 4t^3 + 2\left(t^3 - \frac{5}{3}\right)3t^2 = 0$

for maximum or minimum, t = 0 or 1

Now, $\frac{d^2K}{dt^2} = 12t^2 + 30t^4 - 20t$

$$\left.\frac{d^2K}{dt^2}\right|_{t=0} = 0$$

But, $\left.\dfrac{d^3K}{dt^3}\right|_{t=0} \neq 0 \Rightarrow$ There is neither maxima nor minima at t = 0.

Also, $\dfrac{d^2K}{dt^2} > 0$ at t = 1 $\Rightarrow$ L^2 is minimum at t = 1 i.e., L is minimum at t = 1

So, shortest distance = (value of L at t = 1) – (Radius of the circle) = $\dfrac{\sqrt{13}}{3} - 1$

Example 12 :

Find all possible values of the parameter 'a' so that $x^3 - 3x + a = 0$ has three real and distinct roots.

Solution:

Let $f(x) = x^3 - 3x + a$

$\Rightarrow \quad f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$

Clearly x = –1 is the point of maxima and x = 1 is the point of minima.

Now, f(1) = a – 2, f(–1) = a + 2

The roots of f(x) = 0 would be real and distinct if f(1) f(–1) < 0

$\Rightarrow \quad (a - 2)(a + 2) < 0 \Rightarrow -2 < a < 2$

$\Rightarrow$ Thus given equation would have real and distinct roots if $a \in (-2, 2)$.

SOLVED COMPREHENSIVE PASSAGE

Example 13 :

If $f(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$ and $g(x) = x(1 - x^2)$ and h(x) is such that $h''(x) = 6x - 4$ and a local minimum value is 5 at x = 1. Then

1. Range of $\sin^{-1}\sqrt{(fog(x))}$ is

(a) $\left(0, \dfrac{\pi}{2}\right)$ (b) $\left\{0, \dfrac{\pi}{2}\right\}$

(c) $\left\{-\dfrac{\pi}{2}, 0, \dfrac{\pi}{2}\right\}$ (d) $\left\{\dfrac{\pi}{2}\right\}$

2. Area bounded by y = h(x), y = g(f(x)) between x = 0 and x = 2

(a) $\frac{23}{3}$ sq. units (b) $\frac{20}{3}$ sq. units

(c) $\frac{32}{3}$ sq. units (d) $\frac{40}{3}$ sq. units

3. Range of h(x) is $\forall$ x $\in$ [0, 3]

(a) [5, 17] (b) $\left[\frac{139}{27}, 17\right]$

(c) [5, 17] (d) $\left[-\frac{139}{27}, 17\right]$

4. Equation of tangent at (1, 5) to the curve y = h(x) is

(a) x = 1 (b) y = 5

(c) y = 0 (d) x = 0

***Solution*:**

$$f(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases} \quad g(x) = x(1 - x^2)$$

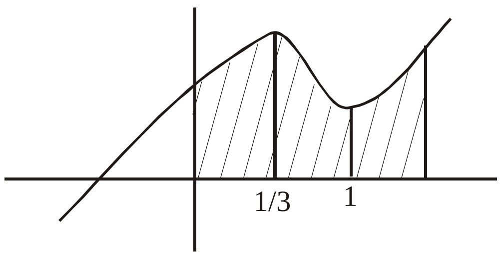
$$fog(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$$

gof(x) = 0 x $\in$ R

1. $\sqrt{fog(x)} = \{0, 1\}$

$\Rightarrow$ Range of $\sin^{-1}\sqrt{fog(x)} = \left\{0, \frac{\pi}{2}\right\}$

2. $h''(x) = 6x - 4$ $h'(x) = 3x^2 - 4x + c$ as $h'(x) = 0$ $\Rightarrow$ c = 1 h(x) = $x^3 - 2x^2 + x + k$ as h(1) = 5 $\Rightarrow$ k = 5 h (x) = $x^3 - 2x^2 + x + 5$ $h'(x)$ = (3x – 1) (x – 1) Area bounded by y = h(x) = $x^3 - 2x^2 + x + 5$ y = gof(x) = 0



req. area = $\int_0^2 h(x)dx = \left[\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2} + 5x\right]_0^2$

$= 4 - \frac{16}{3} + 2 + 10 = -\frac{4}{3} + 12 = \frac{32}{3}$ sq. units

3. h(0) = 5 h (1) = 5

h(3) = 17 $h\left(\frac{1}{3}\right) = \frac{139}{27}$

Range is [5, 17]

4. As x = 1 is point of y = h(x) so tangent is parallel to x–axis $\Rightarrow$ y = 5 is req. tangent

Example 14 :

For $x \in [0, 2\pi]$, the tangent function always increases (however it is discontinuous at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$), the cosine function decreases for $x \in [0, \pi]$ and increases for $x \in [\pi, 2\pi]$ and the sine function decreases for $x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ and increases else where in $[0, 2\pi]$. It is a known fact that (fog) (x) increases if and only if both f and g either decrease or increase and fog (x) decreases otherwise.

5. Among sin (tan 2), $\sin\left(\tan\frac{5}{2}\right)$, sin (tan 3) and sin (tan 4), which one is smallest?

(a) sin (tan 2) (b) sin
(c) sin (tan 3) (d) sin (tan 4)

6. tan (cos x) increases in

(a) (b)
(c) (d) none of these

7. Which of the following is biggest?

(a) cos 1 (b) cos 2
(c) cos 3 (d) cos 4

8. Which of the following is correct?

(a) there is no solution for sin (sin x) = cos (sin x), $x \in$
(b) $x_1 < x_2 \Rightarrow \tan x_1 < \tan x_2$
(c) the maximum value of tan (tan x) is attained at some $x \in [0, \pi/2)$
(d) none of these

9. Which of the following is a decreasing function in ?

(a) sin (sin (cos x)) (b) cos (cos (tan x))
(c) sin (cos(tan x)) (d) sin (cos (sin x))

***Solution*:**

5. $2, \frac{5}{2}, 3 \in \left(\frac{\pi}{2}, \pi\right)$, in which sine function decreases and tangent function increases. Hence the function sin (tan x) is a decreasing function in $\left(\frac{\pi}{2}, \pi\right)$. Thus sin (tan 3) is smallest among first three numbers. Now sin(tan 3) < 0 and sin (tan 4) > 0. Thus sin (tan 3) is the smallest.

6. Obviously (3) is the correct answer.

7. Among cos 1, cos 2, cos 3; cos 1 is biggest as cosine function decreases in [0, π]. As cos 4 < 0, cos 1 is the biggest.

8. sin (sin x) increases from 0 to sin 1, while cos (sin x) decreses from 1 to cos 1. As $[0, \sin 1] \cap [\cos 1, 1] \neq \phi$, there is a solution for sin(sinx) = cos(sinx), $x \in \left(0, \frac{\pi}{2}\right)$. Further $x_1 < x_2 \Rightarrow \tan x_1 < \tan x_2$ is not always true, as tangent function is discontinuous. $\lim_{\tan x \to \frac{\pi}{2}^-} \tan(\tan x) = \infty$, the maximum value does not exist.

9. Obviously (3) is the correct choice.

***Example* 15 :**

If $f(x) = ax^3 + bx^2 + cx + d$ has 3 distinct real roots then $f'(x) = 0$ has two distinct roots and $f(\alpha) f(\beta) < 0$, where $f'(\alpha) = f'(\beta) = 0$

10. The number of real roots of the equation $x^3 - 3x + 1 = 0$ is

(a) 0 (b) 1
(c) 2 (d) 3

11. $x^3 + ax + b = 0$ has three real roots if

(a) $4a^3 + 27b^2 < 0$ (b) $4a^3 + 27b^2 > 0$
(c) $4a^3 - 27b^2 < 0$ (d) none of these

***Solution*:**

10. $f(x) = x^3 - 3x + 1$

$f'(x) = 3(x^2 - 1) = 0$

$f(-1) f(1) < 0$

11. $f(x) = x^3 + ax + b \Rightarrow f'(x) = 0 \Rightarrow x = \pm\sqrt{-\frac{a}{3}} \Rightarrow f\left(\sqrt{\frac{-a}{3}}\right) f\left(\sqrt{\frac{-a}{3}}\right) < 0$

MATCH THE FOLLOWING

1. Let the function defined in column 1 have domain $(0, \pi)$

	Column I	***Column II***
(a)	$x^2 + 2\cos x$	(P) increasing
(b)	$8\log x + x + 16/x$	(Q) decreasing
(c)	$\log\left(\sqrt{1+x^2} - x\right)$	(R) neither increasing nor decreasing

Ans. a – p, b – r, c – q

Solution :

Let $f(x) = x^2 + 2\cos x$ so $f'(x) = 2(x - \sin x)$ since $x \geq \sin x$ on $(0, \pi)$ so f is increasing.

If $f(x) = 8\log x + x + \frac{16}{x}$ then $f'(x) = \frac{8}{x} + 1 - \frac{16}{x^2} = \frac{x^2 + 8x - 16}{x^2}$

$$= \frac{\left(x + \left(4 + 4\sqrt{2}\right)\right)\left(x + 4 - 4\sqrt{2}\right)}{x^2}$$

Hence f neither increases nor decreases.

If $f(x) = \log\left(\sqrt{1+x^2} - x\right)$ then $f'(x) = -\frac{1}{\sqrt{1+x^2}} < 0$ for all x.

SOLVED OBJECTIVE EXAMPLES

***Example* 1 :**

If the line ax + by + c = 0 is normal to the curve xy + 5 = 0, then

(A) a > 0 , b > 0 (B) b > 0, a < 0

(C) b < 0, a > 0 (D) none of these

***Solution*:**

$xy = -5 \Rightarrow x\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x} > 0$ (as xy = – 5 < 0)

$\Rightarrow$ The slope of the normal is negative

$\Rightarrow -\frac{a}{b} < 0 \Rightarrow \frac{a}{b} > 0 \Rightarrow a > 0, \ b > 0$ or $a < 0, b < 0$

Hence (A) is correct answer.

***Example* 2 :**

The maximum value of $\frac{\tan^2 x - \cot^2 x + 1}{\tan^2 x + \cot^2 x + 1}$ is

(A) $\frac{3}{2}$ (B) 3

(C) $\frac{5}{2}$ (D) none of these

***Solution*:**

Let $y = \frac{\tan^2 x - \cot^2 x + 1}{\tan^2 x + \cot^2 x + 1} = \frac{\tan^4 x + \tan^2 x - 1}{\tan^4 x + \tan^2 x + 1}$

$\Rightarrow (y - 1)\tan^4 x + (y - 1)\tan^2 x + (y + 1) = 0$

$\Rightarrow (y - 1)^2 - 4(y^2 - 1) \geq 0$

$\Rightarrow -3y^2 - 2y + 5 \geq 0 \Rightarrow 3y^2 + 2y - 5 \leq 0$

$\Rightarrow (3y + 5)(y - 1) \leq 0 \Rightarrow -\frac{5}{3} \leq y \leq 1$

$\therefore$ Max. y = 1.

***Example* 3 :**

If S is the set, such that $f(x) = 8x^2 - \ln|x|$ increases in S, then S contains

(A) $\left[-\frac{1}{4}, 0\right)$ (B) $(-\infty, -\frac{1}{4})$

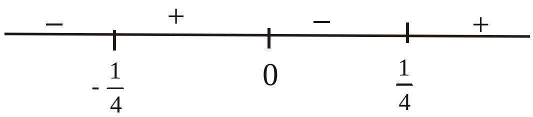
(C) $\left(0, \frac{1}{4}\right)$ (D) none of these

***Solution*:**

$$f'(x) = 16x - \frac{1}{x} = \frac{16}{x}\left(x^2 - \frac{1}{16}\right)$$

For an increasing function. $f'(x) \geq 0 \Rightarrow \frac{1}{x}\left(x - \frac{1}{4}\right)\left(x + \frac{1}{4}\right) \geq 0$

Sign scheme for $f'(x)$:



$$\Rightarrow x \in \left[-\frac{1}{4}, 0\right) \cup \left[\frac{1}{4}, \infty\right)$$

Hence (A) is the correct answer.

***Example* 4 :**

The maximum value of f(x) = |x ln x| in $x \in (0, 1)$ is

(A) 1/e (B) e

(C) 1 (D) none of these

***Solution*:**

f(x) = |x lnx|. for, $x \in (0, 1)$, f(x) = –x lnx

$$f'(x) = -\left[x.\frac{1}{x} + \ln x\right] = -[1 + \ln x] = 0 \quad \Rightarrow x = 1/e$$

$f''(x) = [-1/x] < 0$

f(x) will be maximum at x = 1/e

Maximum value of f(x) = | –1/e | = 1/e

Hence (A) is the correct answer.

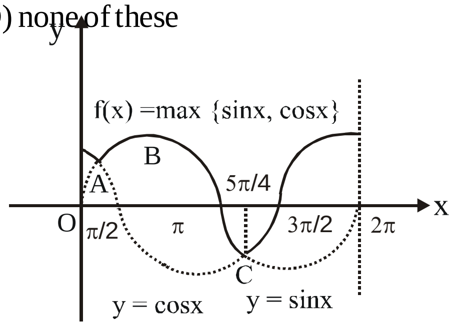
***Example* 5 :**

The number of critical points of f(x) = max {sinx, cosx} for $x \in (0, 2\pi)$ is

(A) 2 (B) 5

(C) 3 (D) none of these

***Solution*:**



Clearly A, B and C are the critical points

Hence (C) is the correct answer.

***Example* 6 :**

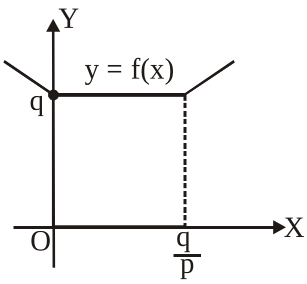
The function $f(x) = |px - q| + r|x|, x \in (-\infty, \infty)$, where p > 0, q > 0, r > 0 assumes its minimum value only at one point if

(A) $p \neq q$ (B) $r \neq q$

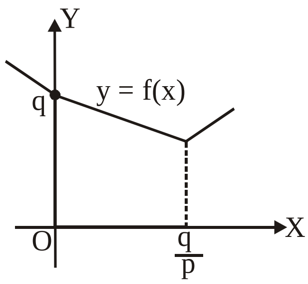
(C) $r \neq p$ (D) $p = q = r$

***Solution*:**

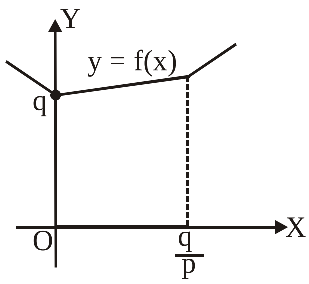
$$f(x) = -px + q - rx,\ x \leq 0 = -px + q + rx,\ 0 < x < \frac{q}{p} = px + q + rx, \frac{q}{p} < x$$



when r = p



when r = p



when r = p

Thus, f has two points of minimum if r = p.

In case $p \neq r$, then x = 0 is point of minimum if r > p and $x = \frac{q}{p}$ is point of minimum if r < p.

***Example 7* :**

The function $f(x) = \frac{x^2 - 3x + 2}{x^2 + 2x - 3}$

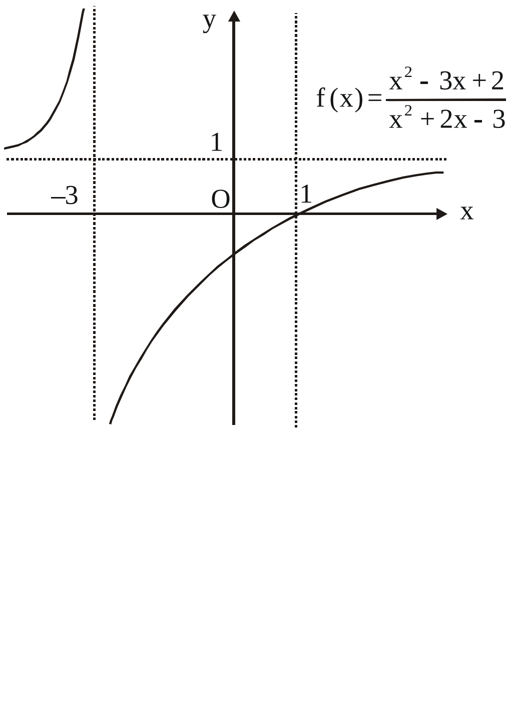
(A) has a maximum value at x = –3

(B) has a minimum value at x = 3 and maximum value at x = 1

(C) is increasing in its domain (D) none of these

***Solution*:**

$$f(x) = \frac{x^2 - 3x + 2}{x^2 + 2x - 3} = \frac{(x-1)(x-2)}{(x-1)(x+3)} = \frac{x-2}{x+3},\quad x \neq 1, -3$$



$$\frac{df(x)}{dx}=\frac{(x+3)-(x-2)}{(x+3)^2}=\frac{5}{(x+3)^2}>0\forall x\neq 1,-3$$

Clearly f(x) is increasing in its domain, Hence (C) is the correct answer.

***Example* 8 :**

Least natural number a for which $x + ax^{-2} > 2 \ \forall x \in (0,\infty)$ is

(A) 1 (B) 2

(C) 5 (D) none of these

***Solution*:**

Let $f(x) = x + ax^{-2}$

$$f'(x)=1-2ax^3=0 \Rightarrow x=(2a)^{-1/3}$$

$$f''(x) = 6ax^4 > 0 \ \forall x \in (0,\infty) \text{ (as a is a natural number)}$$

Thus $(2a)^{1/3} + a(2a)^{-2/3} > 2 \Rightarrow a > \frac{32}{27} \Rightarrow$ least natural number a = 2.

Alternative solution : $x + ax^{-2} > 2 \Rightarrow x^2 - 2x^2 + a > 0$

Let $f(x) = x^3 - 2x^2 + a$

Since $f(x) > 0 \ \forall x \in (0,\infty) \Rightarrow \min f(x) > 0$

For minimum f(x), $f'(x)=3x^2-4x=0 \Rightarrow x=0,\ 4/3$

$$f(4/3) > 0 \Rightarrow a > \frac{32}{27}.$$

Hence (B) is the correct answer.

***Example* 9 :**

If $f(x) = (\sin^2 x - 1)^n (2 + \cos^2 x)$, then $x = \pi/2$ is a point of

(A) local maximum, if n is odd (B) local minimum if n is odd

(C) local maximum, if n is even (D) none of these

***Solution*:**

If x = a is the point of local extremum of

y = f(x), then $f(a-h).\ f(a+h) > 0 \Rightarrow f(\pi/2-h)\ .\ f(\pi/2+h) > 0$

$$(f(\pi/2-h)=(-ve)^n \quad \text{... (1)}$$

$$f(\pi/2+h)=(-ve)^n \quad \text{... (2)}$$

$$f(\pi/2)=0 \quad \text{... (3)}$$

$\Rightarrow \quad f(\pi/2-h).\ f(\pi/2+h) = (-ve)^{2n} > 0 \Rightarrow$ n can be odd or even.

So from (1), (2) and (3), if n is odd or even maxima or minima occurs accordingly.

Hence (A) is the correct answers.

Example 10 :

Let N be any four digit number say $x_1\, x_2\, x_3\, x_4$. Then maximum value of $\dfrac{N}{x_1+x_2+x_3+x_4}$ is equal to

(A) 1000 (B) $\dfrac{1111}{4}$

(C) 800 (D) none of these

Solution:

$$\frac{N}{x_1+x_2+x_3+x_4} = \frac{1000x_1+100x_2+10x_3+x_4}{x_1+x_2+x_3+x_4} = 1000 - \frac{(900x_2+990x_3+999x_4)}{(x_1+x_2+x_3+x_4)}$$

$\Rightarrow \quad$ maximum value of $\dfrac{N}{x_1+x_2+x_3+x_4} = 1000$

Hence (A) is the correct answer.

Example 11 :

Let $f(x) = \begin{cases} |x-1|+a, & x<1 \\ 2x+3, & x \ge 1 \end{cases}$. If f(x) has a local minima at x = 1, then

(A) $a = 5$ (B) $a < 5$

(C) $a > 5$ (D) none of these

Solution:

$$f(x) = \begin{cases} 1-x+a, & x<1 \\ 2x+3, & x\ge 1 \end{cases}$$

Local minimum value of f(x) at x = 1, will be 5

i.e., $1 - x + a \geq 5$ at $x = 1$ or, $a \geq 5$.

Hence (A) is the correct answer.

Example 12 :

If $f(x) = a_0 + a_1 x^2 + a_2 x^4 + \ldots + a_n x^{2n}$ be a polynomial where $a_0 < a_1 < a_2 < \ldots < a_n$ and all are positive then f(x) has

(A) neither a maximum nor a minimum (B) only one maximum

(C) only one minimum (D) none of these

Solution:

$$f'(x) = 2a_1 x + 4a_2 x^3 + \ldots.. + 2n\, a_n x^{2n-1}$$

$= 2x\,(a_1 + 2a_2\,x^2 + \ldots.. + na_n\,x^{2n-2})$

As $(a_1 + 2a_2x^2 + \ldots.. + n\,a_n\,x^{2n-2}) > 0\ \ \forall x \in R$

$\Rightarrow$ $f'(x) > 0,\ x > 0$ and $f'(x) < 0,\ x < 0$ $\Rightarrow$ $x = 0$ is the only point of minima.

Hence (C) is the correct answer.

***Example* 13 :**

If $2a + 3b + 6c = 0$, then the equation $ax^2 + bx + c = 0$ will have atleast one root in

(A) $(-2, -1)$ (B) $(-1, 0)$

(C) $(0, 1)$ (D) $(1, 2)$

***Solution*:**

Let $f(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx$ which is a differentiable function

$f(0) = 0,\ f(1) = \frac{a}{3} + \frac{b}{2} + c = \frac{(2a+3b+6c)}{6} = 0$ (given)

So, according to Rolle's theorem

$f'(c) = 0$ for atleast one $c \in (0,1)$

Hence (C) is the correct answer.

***Example* 14 :**

Let $f(x) = (1 + b^2)x^2 + 2bx + 1$ and let m(B) be the minimum value of f(x). As b varies, the range of m(B) is

(A) $[0, 1]$ (B) $\left(0, \frac{1}{2}\right]$

(C) $\left[\frac{1}{2}, 1\right]$ (D) $(0, 1]$

***Solution*:**

$f(x) = (1 + b^2)x^2 + 2bx + 1$

$f'(x) = (1 + b^2)2x + 2b$

$f''(x) = (1 + b^2)2 > 0$ for all x

$\therefore$ f(x) is minimum when $f'(x) = 0$

i.e. when $(1 + b^2)2x + 2b = 0$

i.e. $x = -\frac{b}{1+b^2}$

$\therefore$ Min. value of

$$f(x) = (1+b^2).\frac{b^2}{(1+b^2)^2} - \frac{2b^2}{1+b^2} + 1$$

$$= \frac{b^2 - 2b^2 + 1 + b^2}{1 + b^2} = \frac{1}{1 + b^2}$$

$$\therefore \quad m(b) = \frac{1}{1 + b^2}$$

since $\frac{1}{1 + b^2} \le 1$

$$[\because 1 + b^2 \ge 0 \text{ for all } b \in R]$$

and $\frac{1}{1 + b^2} > 0 \ \forall b \in R$

$$\therefore \ 0 < \frac{1}{1 + b^2} \le 1 \Rightarrow 0 < m(b) \le 1$$

$\therefore$ Range of m(B) = (0, 1].

***Example* 15 :**

If α is the root (having least absolute value) of the equation $x^2 - bx - 1 = 0$ ($b \in R^+$) then

(A) $\alpha < -1$ (B) $-1 < \alpha < 0$

(C) $0 < \alpha < 1$ (D) $\alpha > 1$

***Solution*:**

Let $f(x) = x^2 - bx - 1$ $(b \in R^+)$

$f(-1) = b = +$ ve

$f(0) = -1 = -$ve

$f(1) = -b = -$ve

Clearly one root lies in $(-1, 0)$ and other in $(1, \infty)$

So, α (having least absolute value) $\in (-1, 0)$.

Hence (B) is the correct answer.