SOLVED SUBJECTIVE EXAMPLES

***Example* 1 :**

Evaluate $I = \int_0^{\pi} \frac{x\,dx}{(a^2\cos^2 x + b^2 \sin^2 x)^2}$

***Solution*:**

Applying $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, $I = \int_0^{\pi} \frac{(\pi - x)dx}{(a^2\cos^2 x + b^2\sin^2 x)^2}$

adding, $2I = \int_0^{\pi} \frac{\pi dx}{(a^2\cos^2 x + b^2\sin^2 x)^2}$

Here f(2a–x) = f(x). Thus, we have

$$2I = 2\int_0^{\pi/2} \frac{\pi dx}{(a^2\cos^2 x + b^2\sin^2 x)^2}$$

or $$I = \pi\int_0^{\pi/2} \frac{dx}{(a^2\cos^2 x + b^2\sin^2 x)^2} = \pi\int_0^{\pi/2} \frac{\sec^4 x\,dx}{(a^2 + b^2\tan^2 x)^2}$$

Put b tanx = a tanθ, b$\sec^2$xdx = a$\sec^2\theta d\theta$

$$I = \pi\int_0^{\pi/2} \frac{(1 + \frac{a^2}{b^2} + \tan^2\theta)\frac{a}{b}\sec^2\theta\,d\theta}{a^{2+}(1+\tan^2\theta)^2}$$

$$= \pi\int_0^{\pi/2} \frac{a(1+\frac{a^2}{b^2}\tan^2\theta)}{ba^4\sec^2\theta}d\theta$$

$$= \frac{\pi}{a^3 b}\int_0^{\pi/2}(1+\frac{a^2}{b^2}\tan^2\theta)\cos^2\theta d\theta = \frac{\pi}{a^3b^2}\int_0^{\pi/2}(b^2\cos^2\theta + a^2\sin^2\theta)d\theta = \frac{\pi}{a^3b^2}I_1 \text{ (say)}$$

Now $I_1 = \int_0^{\pi/2}(b^2\cos^2\theta + a^2\sin^2\theta)d\theta$

Applying $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$,

$$= \int_0^{\pi/2}(b^2\sin^2\theta + a^2\cos^2\theta)d\theta$$

adding $2I_1 = \int_0^{\pi/2} (a^2 + b^2) d\theta = \frac{\pi}{2}(a^2 + b^2)$ or $I_1 = \frac{\pi}{4}(a^2 + b^2)$

$\therefore \quad I = \frac{\pi}{a^3 b^3}\left(\frac{\pi}{4}\right)(a^2 + b^2) = \frac{\pi^2}{4a^3b^3}(a^2 + b^2)$

Example 2 :

Evaluate $\int_0^{\pi} \frac{x\,dx}{1+\cos^2 x}$

***Solution*:**

Let $I = \int_0^{\pi} \frac{x}{1+\cos^2 x} dx$

$$\int_0^{\pi} \frac{(\pi - x)}{1+\cos^2(\pi - x)} dx = \int_0^{\pi} \frac{\pi\,dx}{1+\cos^2 x} - \int_0^{\pi} \frac{x\,dx}{1+\cos^2 x}$$

$$I = \pi \int_0^{\pi} \frac{dx}{1+\cos^2 x} - I$$

$$2I = \pi \int_0^{\pi} \frac{dx}{1+\cos^2 x} = 2\pi \int_0^{\pi/2} \frac{dx}{1+\cos^2 x} \quad \text{(as } f(2a - x) = f(x))$$

$$\Rightarrow \quad I = \pi \int_0^{\pi/2} \frac{\sec^2 x}{1+\sec^2 x} dx$$

Let $\tan x = t$, $\sec^2 x\, dx = dt$

$$\therefore \quad I = \pi \int_0^{\infty} \frac{dt}{2+t^2} = \frac{\pi}{\sqrt{2}}\left[\tan^{-1}\frac{t}{\sqrt{2}}\right]_0^{\infty}$$

$$I = \frac{\pi}{\sqrt{2}}\left(\frac{\pi}{2} - 0\right) = \frac{\pi^2}{2\sqrt{2}}$$

$$\therefore \quad I = \frac{\pi^2}{2\sqrt{2}}$$

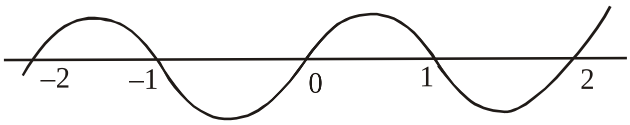
Example 3 :

Find the points of maxima/minima of $\int_0^{x^2} \frac{t^2 - 5t + 4}{2 + e^t} dt$.

***Solution*:**

Let $f(x) = \int_0^{x^2} \frac{t^2 - 5t + 4}{2 + e^t} dt$

$\Rightarrow \; f'(x) = \frac{x^4 - 5x^2 + 4}{2 + e^{x^2}} 2x - 0$



$= \frac{(x-1)(x+1)(x-2)(x+2)\,2x}{2 + e^{x^2}}$

From the wavy curve, it is clear that $f'(x)$ changes its sign at x = ±2, ±1, 0 and hence the points of maxima are –1, 1 and that of the minima are –2, 0, 2.

***Example 4* :**

Evaluate $\int_1^4 (\{x\})^{[x]} dx,$ where {.} and [.] denote the fractional part and the greatest integer functions respectively.

***Solution*:**

$$I = \int_1^4 (\{x\})^{[x]} dx$$

$$= \int_1^4 (x - [x])^{[x]} dx$$

$$= \int_1^2 (x - [x])^{[x]} dx + \int_2^3 (x - [x])^{[x]} dx + \int_3^4 (x - [x])^{[x]} dx$$

$$= \int_1^2 (x-1)^1 dx + \int_2^3 (x-2)^2 dx + \int_3^4 (x-3)^3 dx$$

$$= \left[\frac{(x-1)^2}{2}\right]_1^2 + \left[\frac{(x-2)^3}{3}\right]_2^3 + \left[\frac{(x-3)^4}{4}\right]_3^4$$

$$\left[\frac{1}{2} - 0\right] + \left[\frac{1}{3} - 0\right] + \left[\frac{1}{4} - 0\right] = \frac{13}{12}$$

***Example 5* :**

If $\int_{\pi/3}^{x} \sqrt{3 - \sin^2 t}\; dt + \int_0^y \cos t \; dt = 0$, then evaluate $\frac{dy}{dx}$.

***Solution*:**

Differentiating the given equation w.r.t. to x, we get

$$\frac{d}{dx}\left[\int_{\pi/3}^{x}\sqrt{3-\sin^2 t}\,dt\right]+\frac{d}{dx}\left[\int_{0}^{y}\cos t.dt\right]=0$$

$$\Rightarrow \quad \sqrt{3-\sin^2 x}+\cos y\frac{dy}{dx}=0$$

$$\therefore \quad \frac{dy}{dx}=-\frac{\sqrt{3-\sin^2 x}}{\cos y}$$

***Example* 6 :**

Compute the integral $\int_{-\sqrt{2}}^{\sqrt{2}}\frac{2x^7+3x^6-10x^5-7x^3-12x^2+x+1}{x^2+2}dx$

***Solution*:**

Break the integrand as the sum of two functions, one being even and the other being odd.

$$I=\int_{-\sqrt{2}}^{\sqrt{2}}\frac{3x^6-12x^2+1}{x^2+2}dx+\int_{-\sqrt{2}}^{\sqrt{2}}\frac{2x^7-10x^5-7x^3+x}{x^2+2}dx$$

Now using property

$\int_{-a}^{a}f(x)\,dx=0$, if f(x) is odd and $2\int_{0}^{a}f(x)\,dx$, if f(x) is even

$$\therefore \quad I=2\int_{0}^{\sqrt{2}}\frac{3x^6-12x^2+1}{x^2+2}dx+0=2\int_{0}^{\sqrt{2}}\frac{3x^2(x^4-4)+1}{x^2+2}dx$$

$$=2\int_{0}^{\sqrt{2}}3x^2(x^2-2)dx+2\int_{0}^{\sqrt{2}}\frac{dx}{x^2+2}$$

$$=\left|\left(\frac{6x^5}{5}-4x^3\right)\right|_{0}^{\sqrt{2}}+\frac{2}{\sqrt{2}}\left|\tan^{-1}\frac{x}{\sqrt{2}}\right|_{0}^{\sqrt{2}}$$

$$=\frac{\pi}{2\sqrt{2}}-\frac{16}{5}\sqrt{2}$$

***Example* 7 :**

Prove that $\int_{0}^{\pi/4}\frac{x^2}{(x\sin x+\cos x)^2}dx=\frac{4-\pi}{4+\pi}$.

***Solution*:**

$$\int_0^{\pi/4} \frac{x\cos x}{(x\sin x+\cos x)^2}.x\sec x\,dx$$

$$=\left[-\frac{x\sec x}{x\sin x+\cos x}\right]_0^{\pi/4}+\int_0^{\pi/4}\frac{\sec x+x\sec x.\tan x}{(x\sin x+\cos x)}dx$$

$$=-\frac{(\pi/4)\sqrt{2}.\sqrt{2}}{\left(\frac{\pi}{4}\right)+1}+\int_0^{\pi/4}\frac{1}{\cos^2 x}.\frac{\cos x+x\sin x}{x\sin x+\cos x}dx$$

$$=\frac{-2\pi}{\pi+4}+\int_0^{\pi/4}\sec^2 x dx=-\frac{2\pi}{\pi+4}+[\tan x]_0^{\pi/4} \quad =-\frac{2\pi}{\pi+4}+1=\frac{4-\pi}{4+\pi}$$

***Illustration* 8 :**

Evaluate $\int_0^{\pi/4} \ln(1+\tan x)\ dx$ **[IIT- 1997C]**

***Solution*:**

Let f(x) = ln (1 + tanx)

$$f\left(\frac{\pi}{4}-x\right) = \ln\left[1+\tan\left(\frac{\pi}{4}-x\right)\right] = \ln\left[1+\frac{1-\tan x}{1+\tan x}\right] = \ln\frac{2}{1+\tan x}$$

$$\therefore \quad f(x)+f\left(\frac{\pi}{4}-x\right)=\ln(1+\tan x)+\ln\frac{2}{1+\tan x}=\ln 2$$

$$\therefore \quad I=\frac{1}{2}\int_0^{\pi/4}\left(f(x)+f\left(\frac{\pi}{4}-x\right)\right)dx=\frac{1}{2}\int_0^{\pi/4}\ln 2dx$$

$$=\frac{1}{2}\,\frac{\pi}{4}\ln 2=\frac{\pi}{8}\ln 2$$

***Example* 9 :**

Evaluate $\int_0^1 \cot^{-1}(1+x^2-x)\,dx$ **[IIT- 1998-8M]**

***Solution*:**

$$\text{Let } I=\int_0^1\cot^{-1}(1+x^2-x)=\int_0^1\tan^{-1}\frac{1}{1+x^2-x}dx$$

$$=\int_0^1\tan^{-1}\frac{1}{1+x(x-1)}dx=\int_0^1\tan^{-1}\left\{\frac{x-(x-1)}{1+x(x-1)}\right\}dx$$

$$= \int_0^1 \tan^{-1} x dx - \int_0^1 \tan^{-1}(x-1)\, dx$$

$$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1} x\, dx \quad = 2\int_0^1 \tan^{-1} x\, dx$$

$$= 2\left[x \tan^{-1} x - \frac{1}{2}\ln(1+x^2)\right]_0^1 \quad = 2\left[\frac{\pi}{4} - \frac{\ln 2}{2}\right] \quad = \frac{\pi}{2} - \log 2$$

***Example* 10 :**

Evaluate $\int_0^{2\pi} [\cot^{-1} x] dx$, where [.] denotes the greatest integer function.

***Solution*:**

Let $I = \int_0^{2\pi} \left[\cot^{-1} x\right] dx \quad = \int_0^{\cot 1} [\cot^{-1} x] dx + \int_{\cot 1}^{2\pi} [\cot^{-1} x] dx \quad = \int_0^{\cot 1} 1.dx + \int_{\cot 1}^{2\pi} 0.dx \quad = \cot 1$

***Example* 11 :**

Find $\lim_{n\to\infty}\left[\frac{n+1}{n}\,\frac{n+2}{n}\,...\frac{n+n}{n}\right]^{1/n}$

***Solution*:**

Let $L = \lim_{n\to\infty}\left[\frac{n+1}{n}\,\frac{n+2}{n}...\frac{n+n}{n}\right]^{1/n}$

$$= \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right).....\left(1+\frac{n}{n}\right)\right]^{1/n}$$

$$\ln L = \lim_{n\to\infty}\frac{1}{n}\left[\ln\left(1+\frac{1}{n}\right) + \ln\left(1+\frac{2}{n}\right) + ... + \ln\left(1+\frac{n}{n}\right)\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\ln\left(1+\frac{r}{n}\right) \quad = \int_0^1 \ln(1+x)\, dx$$

$$= [x\ln(1+x)]_0^1 - \int_0^1 \frac{x}{x+1} dx \quad = \ln 2 - \int_0^1 \left(1 - \frac{1}{1+x}\right) dx$$

$\Rightarrow \quad L = 4/e$

***Example* 12 :**

Prove that $\int_0^x e^{xt}.e^{-t^2}\,dt = e^{x^2/4}\int_0^x e^{-t^2/4}dt$

***Solution*:**

Let $I = \int_0^x e^{xt}.e^{-t^2}dt$

Let $t = \dfrac{x+z}{2}$, $dt = \dfrac{1}{2}dz$ [we want to convert $t(x-t)$ to $\left(\dfrac{x+z}{2}\right)\left(\dfrac{x-z}{2}\right)$]

$$I = \int_0^x e^{tx}.e^{-t^2}\,dt = \frac{1}{2}\int_{-x}^{x} e^{\frac{(x+z)}{2}.x}\, e^{-\left(\frac{x+z}{2}\right)^2}\, dz \quad = \frac{1}{2}\int_{-x}^{x} e^{\frac{x^2}{4}-\frac{z^2}{4}}\, dz$$

$$= \frac{1}{2}e^{\frac{x^2}{4}}\int_{-x}^{x} e^{-\frac{z^2}{4}}dz = \frac{1}{2}e^{x^2/4}\int_{-x}^{x} e^{-\frac{t^2}{4}}\,dt \; = \frac{1}{2}e^{x^2/4}\, 2\int_0^x e^{-t^2/4}\,dt \; = e^{x^2/4}\int_0^x e^{-t^2/4}\,dt$$

$$\therefore \quad \int_0^x e^{xt}.e^{-t^2}dt = e^{x^2/4}\int_0^x e^{-t^2/4}\,dt$$

***Example* 13 :**

Evaluate $\int_0^\infty \ln\left(x + \dfrac{1}{x}\right).\,\dfrac{dx}{1+x^2}$

***Solution*:**

Let $I = \int_0^\infty \ln\left(x + \dfrac{1}{x}\right).\,\dfrac{dx}{1+x^2}$

Let $x = \tan\theta \;\Rightarrow\; dx = \sec^2\theta d\theta$

$$\therefore \quad I = \int_0^{\pi/2} \ln(\tan\theta + \cot\theta)\,\frac{\sec^2\theta}{1+\tan^2\theta}d\theta$$

$$\int_0^{\pi/2} \ln(\tan\theta + \cot\theta)d\theta = \int_0^{\pi/2} \ln\frac{(\sin^2\theta + \cos^2\theta)}{\sin\theta\cos\theta}d\theta$$

$$= -\int_0^{\pi/2} \ln\sin\theta\, d\theta - \int_0^{\pi/2} \ln\cos\theta\, d\theta \; = -\left[-\frac{\pi}{2}\ln 2\right] - \left[-\frac{\pi}{2}\ln 2\right] = \pi\ln 2$$

***Example* 14 :**

Let f be a continuous function on [a, b]. Prove that there exists a number $x \in [a, b]$ such that

$$\int_a^x f(t)\,dt = \int_x^b f(t)\,dt$$

***Solution*:**

Let g(x) = $\int_a^x f(t)\,dt - \int_x^b f(t)\,dt,\ x \in [a, b]$

We have g(A) = $-\int_a^b f(t)\,dt$ and g(B) = $\int_a^b f(t)\,dt \Rightarrow$ g(A). g(B) = $-\left[\int_a^b f(t)\,dt\right]^2 \le 0$

Clearly g(x) is continuous in [a, b] and g(A). g(B) ≤ 0

It implies that g(x) will become zero at least once in [a, b]. Hence $\int_a^x f(t)\,dt = \int_x^b f(t)\,dt$ for at least one value of $x \in [a, b]$.

***Example* 15 :**

Let f(x) be a continuous function such that f(a – x) + f(x) = 0 for $x \in [0, a]$. Find $\int_0^a \frac{dx}{1+e^{f(x)}}$

***Solution*:**

Let I = $\int_0^a \frac{dx}{1+e^{f(x)}} = \int_0^a \frac{dx}{1+e^{f(a-x)}} = \int_0^a \frac{dx}{1+e^{-f(x)}} = \int_0^a \frac{e^{f(x)}}{1+e^{f(x)}}dx$

$$\therefore \quad I + I = \int_0^a \frac{dx}{1+e^{f(x)}} + \int_0^a \frac{e^{f(x)}}{1+e^{f(x)}}dx$$

$$2I = \int_0^a 1.dx = a \quad \Rightarrow \quad I = \frac{a}{2}$$

SOLVED OBJECTIVE EXAMPLES

***Example* 1 :**

If $I = \int_0^{\pi/4} x^2 \cos 2x \, dx$ then

(A) $\frac{\pi^2}{32} + \frac{1}{4}$

(B) $\frac{\pi^2}{32} - \frac{1}{4}$

(C) $\frac{\pi^2}{32} - \frac{1}{8}$

(D) $\frac{\pi^2}{32} + \frac{1}{8}$

***Solution*:**

Integrating by parts

$$I = \left[x^2 \frac{\sin 2x}{2} \right]_0^{\pi/4} - \int_0^{\pi/4} x \sin 2x dx$$

$$= \frac{\pi^2}{32} + \left[x.\frac{\cos 2x}{2} \right]_0^{\pi/4} - \int_0^{\pi/4} \cos \frac{2x}{2} dx \; = \frac{\pi^2}{32} + 0 - \frac{1}{2}\left[\frac{\sin 2x}{2} \right]_0^{\pi/4} = \frac{\pi^2}{32} - \frac{1}{4}$$

***Example* 2 :**

If $I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} dx$ then

(A) $\frac{\pi}{4}$

(B) π

(C) $\frac{\pi}{3}$

(D) $\frac{\pi}{2}$

***Solution*:**

$$I = \int_{-\pi}^{\pi} \frac{\cos^2(-x)}{1+a^{-x}} dx \; : \quad I = \int_{-\pi}^{\pi} \frac{a^x \cos^2 x}{1+a^x} dx$$

adding $2I = \int_{-\pi}^{\pi} \cos^2 x \, dx = 2 \int_0^{\pi} \cos^2 x dx$ [$\because$ $f(x) = \cos^2 x = f(-x)$]

$$= 2\int_0^{\pi} \cos^2 .x dx \Rightarrow \left[x + \frac{\sin 2x}{2} \right]_0^{\pi} = \pi$$

$2I = \pi$

$\therefore \quad I = \pi/2$

Example 3 :

Let f(x) = x – [x], for every real number x, where [x] is integral part of x, then $\int_{-1}^{1} f(x)\,dx$ is

(A) 0 (B) 1

(C) –1 (D) 2

Solution:

Let f(x) = x – [x], $I = \int_{-1}^{1} (x-[x])\,dx$

$$= \int_{-1}^{0} (x-[x])\,dx + \int_{0}^{1} (x-[x])\,dx = \int_{-1}^{0} (x+1)\,dx + \int_{0}^{1} x.dx \;=\; \left[\frac{x^2}{2}+x\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_{0}^{1} = 1$$

Example 4 :

$\int_{0}^{100} [\tan^{-1}(x)]\,dx$ is([.] denotes the greatest integer function).

(A) (100 – tan2) (B) (100 + tan 1)

(C) (100 + tan2) (D) (100 – tan 1)

Solution:

$$\int_{0}^{\tan 1} [\tan^{-1} x]dx + \int_{\tan 1}^{100} [\tan^{-1} x]dx$$

$$= \int_{0}^{\tan 1} 0.dx + \int_{\tan 1}^{100} 1.dx \Rightarrow \quad 100 - \tan 1$$

Example 5 :

If f, g and h be continuous function on [0, a] such that f(a –x) = f(x), g(a – x) = – g(x) and 3h(x) – 4h (a–x) = 5, then $\int_{0}^{a} f(x).\, g(x).\, h(x)\,dx$ =

(A) 0 (B) – 1

(C) 2 (D) 1

Solution:

$$I = \int_{0}^{a} f(a-x)\,g(a-x).h(a-x)\ dx$$

$$7I = 3I + 4I \;=\; \int_{0}^{a} f(x)\,g(x)\,\{3h(x) - 4h(a-x)\}\ dx \;=\; 5\int_{0}^{a} f(x)\,g(x)\,dx = 0$$

$\because$ f(a – x) g(a – x) = – f(x) g(x) $\Rightarrow$ I = 0

Example 6 :

Definite integration of $\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}dx$ is

(A) $\frac{1}{3}$

(B) $\frac{1}{4}$

(C) $\frac{1}{2}$

(D) 0

Solution:

Using property $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

$$I = \int_2^3 \frac{\sqrt{2+3-x}}{\sqrt{5-(2+3-x)}+\sqrt{5-x}}dx = \int_2^3 \frac{\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}dx$$

$$I = \int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}dx \text{ adding}$$

$$2I = \int_2^3 \left(\frac{\sqrt{5-x}}{(\sqrt{x}+\sqrt{5-x}} + \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}} \right)dx = \int_2^3 1.dx = 3-2 = 1$$

I = 1/2

Example 7 :

If $f : R \to R$ is continuous and differentiable function such that

$\int_{-1}^{x} f(t)\,dt + f'''(3)\int_x^0 dt = \int_1^x t^3 dt - f'(1)\int_0^x t^2 dt + f''(2)\int_x^3 t\,dt$ then the value of $f'(4)$ is

(A) 48 – $8f'(1) + f''(2)$

(B) 48 – $8f'(1) - f''(2)$

(C) 48 + $8f'(1) + f''(2)$

(D) none of these

Solution:

$\int_{-1}^{x} f(t)\,dt + f'''(3)\int_x^0 dt = \int_1^x t^3 dt - f'(1)\int_0^x t^2 dt + f''(2)\int_x^3 t.dt$ Differentiable w.r.t. ‘x’ both side

f(x).1 – 0 – $f'''(3)$ = x^3 – $f'(1)$. x^2 – $f''(2)$.x

$\Rightarrow \quad f'(x) = 3x^2 - f'(1).2x - f''(2)$

$\therefore \quad f'(4)$ = 48 – $f'(1)$ 8 – $f''(2)$

Example 8 :

If $f(x) = \begin{cases} 0, & \text{where } x = \dfrac{n}{n+1},\ n = 1, 2, 3... \\ 1, & \text{else where} \end{cases}$

then the value of $\int_0^2 f(x)\,dx$

(A) 1 (B) 0

(C) 2 (D) ∞

Solution:

$$\int_0^2 f(x)\,dx = \int_0^{1/2} 1.dx + \int_{1/2}^{2/3} 1.dx + \int_{2/3}^{3/4} 1.dx + ... \int_{\frac{n-1}{n}}^{\frac{n}{n+1}} 1.dx + ... + \int_1^2 dx$$

$$= \frac{1}{2} + \left[\frac{2}{3} - \frac{1}{2}\right] + \left(\frac{3}{4} - \frac{2}{3}\right) + ... + \left(\frac{n}{n+1} - \frac{n-1}{n}\right)_{\text{as } n \to \infty} + ... + 1 \quad = \frac{n}{n+1} + ... + 1$$

we take $n \to \infty$

$$\int_0^2 f(x)\,dx = 1 + 1 = 2$$

Example 9 :

$\int_{-\pi}^{\pi} (\cos px - \sin qx)^2$ dx where p, q are integers is equal to

(A) $-\pi$ (B) 0

(C) π (D) 2π

Solution:

$$I = \int_{-\pi}^{\pi} (\cos^2 px + \sin^2 qx - 2\cos px\ \sin qx)dx$$

$\because$ $\sin^2 qx$, $\cos^2 px$ are even functions of x and cospx .sinqx is an odd function.

$\therefore$ $\int_{-\pi}^{\pi} \cos^2 px\,dx = 2\int_0^{\pi} \cos^2 px dx$

$\int_{-\pi}^{\pi} \sin^2 qx\,dx = 2\int_0^{\pi} \sin^2 qx\,dx$ and $\int_{-\pi}^{\pi} \cos px\ \sin qx\ dx = 0$

$$\therefore \quad I = 2\int_0^{\pi} \cos^2 px\ dx + 2\int_0^{\pi} \sin^2 qx\, dx = 0$$

$$= 2\int_0^{\pi}\left(\frac{1+\cos 2px}{2}\right)dx + 2\int_0^{\pi}\left(\frac{1-\cos 2qx}{2}\right)dx$$

$$= \int_0^{\pi}(1+\cos 2px)\,dx + \int_0^{\pi}(1-\cos 2qx)\,dx = \left[x+\frac{\sin 2px}{2p}\right]_0^{\pi} + \left[x-\frac{\sin 2qx}{2q}\right]_0^{\pi} = 2\pi$$

***Example* 10 :**

The value of integral $\int_0^{\pi/2} \frac{\phi(x)}{\phi(x)+\phi(\pi/2-x)}dx$ is

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{2}$

(C) π (D) none of these

***Solution*:**

Let $I = \int_0^{\pi/2} \frac{\phi(x)}{\phi(x)+\phi\left(\frac{\pi}{2}-x\right)}dx$ then $I = \int_0^{\pi/2} \frac{\phi\left(\frac{\pi}{2}-x\right)}{\phi\left(\frac{\pi}{2}-x\right)+\phi(x)}dx$

adding $2I = \int_0^{\pi/2} \frac{\phi(x)+\phi\left(\frac{\pi}{2}-x\right)}{\phi\left(\frac{\pi}{2}-x\right)+\phi(x)}dx = \int_0^{\pi/2} 1.dx = [x]_0^{\pi/2} = \frac{\pi}{2} \Rightarrow I = \pi/4$

***Example* 11 :**

$\int_0^{\pi} x \sin x \cos^4 x\, dx =$

(A) $\frac{\pi}{10}$ (B) $\frac{\pi}{5}$

(C) $-\frac{\pi}{5}$ (D) none of these

***Solution*:**

$$I = \int_0^{\pi} x \sin x \cos^4 x dx$$

$$= \int_0^{\pi} (\pi - x)\sin(\pi - x)\cos^4(\pi - x)\,dx = \int_0^{\pi} (\pi - x)\sin x\cos^4 x\,dx$$

$$= \pi\int_0^{\pi} \sin x\cos^4 x\ dx - I$$

$$\Rightarrow \qquad 2I = \pi\left[\frac{-\cos^5 x}{5}\right]_0^{\pi} = \frac{\pi}{5}(1+1) = \frac{2\pi}{5}$$

$$I = \frac{\pi}{5}$$

***Example* 12 :**

The value of $\int_{1/e}^{\tan x} \frac{t.dt}{1+t^2} + \int_{1/e}^{\cot x} \frac{dt}{t(1+t^2)}$ is equal to

(A) 1

(B) 1/2

(C) $\pi/4$

(D) none of these

***Solution*:**

$$I(x) = \int_{1/e}^{\tan x} \frac{tdt}{1+t^2} + \int_{1/e}^{\cot x} \frac{dt}{t(1+t^2)}$$

Diff. w.r.t x.

$$\frac{dI(x)}{dx} = \frac{\tan x}{(1+\tan^2 x)}\sec^2 x + \frac{1}{\cot x\,(1+\cot^2 x)} \quad (-\text{cosec}^2 x) = 0$$

I(x) = constant

Let $x = \frac{\pi}{4}$ Thus $I\left(\frac{\pi}{4}\right) = \int_{1/e}^{1} \frac{tdt}{1+t^2} + \int_{1/e}^{1} \frac{dt}{t(1+t^2)} = \int_{1/e}^{t} \frac{t^2+1}{t(t^2+1)}dt = \int_{1/e}^{1} \frac{dt}{t} = [\ln t]_{1/e}^{1} = 1$

***Example* 13 :**

$\lim_{n\to\infty}\left[\frac{1}{na} + \frac{1}{na+1} + \frac{1}{na+2} + \ldots + \frac{1}{nb}\right]$ is equal to

(A) $\ln\left(\frac{b}{a}\right)$

(B) $\ln\left(\frac{a}{b}\right)$

(C) ln a

(D) ln b

***Solution*:**

$$\lim_{n\to\infty}\left[\frac{1}{na} + \frac{1}{na+1} + \ldots + \frac{1}{na+n(b-a)}\right]$$

$$\lim_{n\to\infty}\sum_{r=1}^{n(b-a)}\frac{1}{na+r}=\lim_{n\to\infty}\sum_{r=1}^{n(b-a)}\frac{1}{a+(r/n)}\frac{1}{n} = \int_0^{b-a}\frac{1}{a+x}dx=\ln\left(\frac{b}{a}\right)$$

Example 14:

The values of ‘a’ for which $\int_0^a (3x^2+4x-5)\,dx<a^3-2$ are

(A) $\frac{1}{2}<a<2$

(B) $\frac{1}{2}\le a\le 2$

(C) $a\le\frac{1}{2}$

(D) $a \geq 2$

Solution:

$$\left[\frac{3x^3}{3}+\frac{4x^2}{2}-5x\right]_0^a<a^3-2$$

$\Rightarrow$ $a^3 + 2a^2 - 5a < a^3 - 2$

$\Rightarrow$ $2a^2 - 5a + 2 < 0$

$\Rightarrow$ $2a\,(a-2) - 1\,(a-2) < 0$

$\therefore$ $\frac{1}{2}<a<2$

Example 15:

If $\int_0^\infty e^{-x^2}\,dx=\frac{\sqrt{\pi}}{2}$, then $\int_0^\infty e^{-ax^2}dx$ where a > 0 is

(A) $\frac{\sqrt{\pi}}{2}$

(B) $\frac{\sqrt{\pi}}{2a}$

(C) $2\frac{\sqrt{\pi}}{a}$

(D) $\frac{1}{2}\sqrt{\frac{\pi}{a}}$

Solution:

$I = \int_0^\infty e^{-ax^2}\,dx$ put $\sqrt{a}$ x = z, dx = $\frac{dz}{\sqrt{a}}$

$$I=\int_0^\infty e^{-z^2}\frac{dz}{\sqrt{a}}=\frac{1}{\sqrt{a}}\int_0^\infty e^{-z^2}dz=\frac{\sqrt{\pi}}{2\sqrt{a}}=\frac{1}{2}\sqrt{\frac{\pi}{a}}$$