SOLVED SUBJECTIVE EXAMPLES

Example 1 :

Show that the straight lines whose direction cosines ℓ, m and n are given by the equations $a\ell + bm + cn = 0$, $u\ell^2 + vm^2 + wn^2 = 0$ are perpendicular or parallel according as $a^2(v + w) + b^2(w + u) + c^2(u + v) = 0$ or $\frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w} = 0$.

Solution :

Eliminating ℓ, between the given relations, we have

$$\frac{u(bm + cn)^2}{a^2} + vm^2 + wn^2 = 0$$

$$\Rightarrow \quad (b^2u + a^2v)m^2 + 2ubcmn + (c^2u + a^2w)n^2 = 0 \qquad \text{. . . (i)}$$

If the lines be parallel, their direction cosines are equal so that the two values of m/n must be equal.

The condition for this is

$$u^2b^2c^2 = (b^2u + a^2v)(c^2u + a^2w)$$

$$\Rightarrow \quad \frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w} = 0$$

Again, if ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 be the direction cosines of the two lines then equation (i) gives

$$\frac{m_1}{n_1} \cdot \frac{m_2}{n_2} = \frac{m_1m_2}{n_1n_2} = \frac{c^2u + a^2w}{b^2u + a^2v}$$

$$\Rightarrow \quad \frac{m_1m_2}{c^2u + a^2w} = \frac{n_1n_2}{b^2u + a^2v}$$

Similarly we have,

$$\frac{\ell_1\ell_2}{b^2w + c^2v} = \frac{m_1m_2}{a^2w + c^2u}$$

Thus, we have

$$\frac{\ell_1\ell_2}{b^2w + c^2v} = \frac{m_1m_2}{a^2w + c^2u} = \frac{n_1n_2}{b^2u + a^2v} = k \text{, say}$$

$$\Rightarrow \quad \ell_1\ell_2 + m_1m_2 + n_1n_2 = k(b^2w + c^2v + a^2w + c^2u + b^2u + a^2v)$$

For perpendicular lines

$$\ell_1\ell_2 + m_1m_2 + n_1n_2 = 0$$

Thus, the condition for perpendicularity is

$$a^2(v + w) + b^2(w + u) + c^2(u + v) = 0.$$

Example 2 :

If a variable line in two adjacent positions had direction cosines ℓ, m and n and. Show that the small angle $\delta\theta$ between two positions is given by $\delta\theta^2 = \delta\ell^2 + \delta m^2 + \delta n^2$.

Solution :

Since $[\ell, m, n]$ and $[\ell+\delta\ell, m+\delta m, n+\delta n]$ are dc's hence

$$\ell^2 + m^2 + n^2 = 1 \quad \text{. . . (i)}$$

and $(\ell+\delta\ell)^2 + (m+\delta m)^2 + (n+\delta n)^2 = 1$

$\Rightarrow \quad \delta\ell^2 + \delta m^2 + \delta n^2 = 2(\ell\delta\ell + m\delta m + n\delta n)$ . . . (ii)

Now, $\cos\delta\theta = \ell(\ell+\delta\ell) + m(m+\delta m) + n(n+\delta n)$

$$= \ell^2 + m^2 + n^2 + \ell\delta\ell + m\delta m + n\delta n$$

$$= 1 - \frac{1}{2}\{\delta\ell^2 + \delta m^2 + \delta n^2\} \text{ , from (i) and (ii)}$$

$$= \delta\ell^2 + \delta m^2 + \delta n^2 = 2(1-\cos\delta\theta)$$

$$= 2.2\sin^2\frac{\delta\theta}{2} = 4\left(\frac{1}{2}\delta\theta\right)^2$$

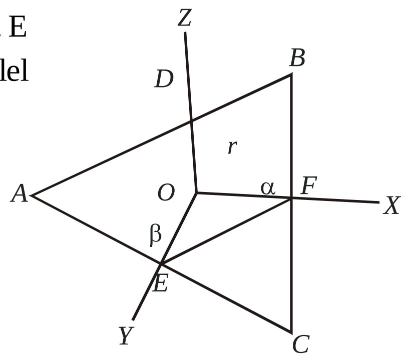
$$= 4\left(\frac{1}{2}\delta\theta\right)^2 = \delta\theta^2 \text{ (as } \delta\theta \text{ is small, } \sin\delta\theta \approx \delta\theta)$$

Example 3 :

A triangle, the lengths of whose sides are a, b and c is placed so that the middle points of the sides are on the axes. Show that the equation to the plane is $\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1$, where

$$\alpha^2 = \frac{(b^2+c^2-a^2)}{8},\ \beta^2 = \frac{(c^2+a^2-b^2)}{8},\ \gamma^2 = \frac{(a^2+b^2-c^2)}{8}.$$

Solution :



Let α, β, γ, be the intercepts of the required plane with the axes. E and F are the mid points of AC and BC. Therefore, EF is parallel and equal to half of AB.

$\because \quad EF^2 = OE^2 + OF^2$

$$= \alpha^2 + \beta^2$$

But

$$EF = \frac{AB}{2} = \frac{c}{2} \quad \Rightarrow \quad \alpha^2 + \beta^2 = \frac{c^2}{4}$$

Similarly, $\beta^2 + \gamma^2 = \frac{a^2}{4}$ and $\gamma^2 + \alpha^2 = \frac{b^2}{4}$

Adding, $\alpha^2 + \beta^2 + \gamma^2 = \frac{a^2+b^2+c^2}{8}$

$\Rightarrow \quad \gamma^2 = \frac{a^2+b^2+c^2}{8} - \frac{c^2}{4} = \frac{a^2+b^2-c^2}{8}$

Similarly, $\alpha^2 = \dfrac{b^2 + c^2 - a^2}{8}$, $\beta^2 = \dfrac{c^2 + a^2 - b^2}{8}$

Hence, the equation of plane is $\dfrac{x}{\alpha} + \dfrac{y}{\beta} + \dfrac{z}{\gamma} = 1$, where $\alpha^2, \beta^2, \lambda^2$ as given above.

Example 4 :

Prove that the four planes my + nz = 0, nz + ℓx = 0, ℓx + my = 0 and $\ell x + my + nz = p$ form a tetrahedron whose volume is $\dfrac{2p^3}{3\ell mn}$.

Solution :

Solving the given equations taking three planes at a time, we get the vertices of the tetrahedron as

O (0, 0, 0), $A\left(-\dfrac{p}{\ell}, \dfrac{p}{m}, \dfrac{p}{n}\right)$, $A\left(\dfrac{p}{\ell}, -\dfrac{p}{m}, \dfrac{p}{n}\right)$ and $C\left(\dfrac{p}{\ell}, \dfrac{p}{m}, -\dfrac{p}{n}\right)$ with these points as vertices, the volume V of the tetrahedron is given by

$$V = \frac{1}{6}\begin{vmatrix} -p/\ell & p/m & p/n \\ p/\ell & -p/m & p/n \\ p/\ell & p/m & -p/n \end{vmatrix} = \frac{p^3}{6\ell mn}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = \frac{2p^3}{3\ell mn}.$$

Example 5 :

Two system of rectangular axes have the same origin. If a plane cuts them at distances a, b, c and a', b', c' respectively from the origin, prove that $\dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2} = \dfrac{1}{a'^2} + \dfrac{1}{b'^2} + \dfrac{1}{c'^2}$.

Solution :

Equations of the plane w.r.t. two systems are

$\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ and $\dfrac{x}{a'} + \dfrac{y}{b'} + \dfrac{z}{c'} = 1$

Since origin is common to both, hence the perpendicular distances of these planes from the origin must be equal. Hence,

$$\text{or} \qquad \frac{1}{\sqrt{\left(\dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2}\right)}} = \frac{1}{\sqrt{\left(\dfrac{1}{a'^2} + \dfrac{1}{b'^2} + \dfrac{1}{c'^2}\right)}}$$

Example 6 :

A variable plane is at a constant distance p from the origin and meets the axes in A, B, C. Through A, B, C, planes are drawn parallel to the coordinate planes. Show that the locus of their point of intersection is $x^{-2} + y^{-2} + z^{-2} = p^{-2}$.

Solution :

Let the variable plane $\ell n + my + nz = p$ cut the axes in A, B, C then

$A = \left(\dfrac{p}{\ell}, 0, 0\right)$, $B = \left(0, \dfrac{p}{m}, 0\right)$, $C = \left(0, 0, \dfrac{p}{n}\right)$

Planes through A, B, C parallel to coordinate planes are $x = \frac{p}{\ell},\ y = \frac{p}{m},\ z = \frac{p}{n}$, and if they intersect in the point (α, β, γ), then $\left(\alpha = \frac{p}{\ell},\ \beta = \frac{p}{m},\ \gamma = \frac{p}{n}\right)$

$$\Rightarrow \quad \left(\ell = \frac{p}{\alpha},\ m = \frac{p}{\beta},\ n = \frac{p}{\gamma}\right)$$

Since, $\ell^2 + m^2 + n^2 = 1$, we have

$$\left(\frac{p}{\alpha}\right)^2 + \left(\frac{p}{\beta}\right)^2 + \left(\frac{p}{\gamma}\right)^2 = 1$$

Hence the locus of (α, β, γ) is

$$p^2(x^{-2} + y^{-2} + z^{-2}) = 1$$

or $\quad x^{-2} + y^{-2} + z^{-2} = p^{-2}$

Example 7 :

Find the equation of the plane through (α, β, γ) and the line $x + py + q = 0 = rz + s$.

Solution :

Any plane through the given line is

$x + py + q + \lambda(rz + s) = 0,\ \lambda \in R$

As required plane passing through (α, β, γ), we have

$\alpha + p\beta + q + \lambda(rz + s) = 0$

$$\Rightarrow \quad \lambda = \frac{\alpha + p\beta + \gamma}{r\gamma + s} \quad .$$

Hence equation of required plane is

$$x + py + q - \left(\frac{\alpha + p\beta + q}{r\gamma + s}\right)(rz + s) = 0$$

Example 8 :

ABC is any triangle and O is any point in the plane of the triangle. AO, BO, CO meet the sides BC, CA, AB in D, E, F respectively. Prove that $\frac{OD}{AD} + \frac{OE}{BE} + \frac{OF}{CF} = 1$.

Solution :

Take O as the origin.

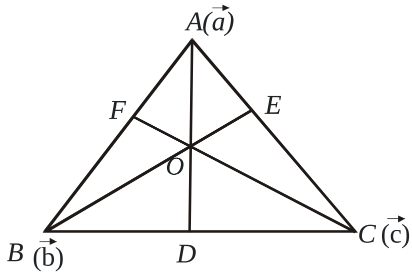
Let $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of the vertices A, B, C of the triangle.

Since $\vec{a}, \vec{b}, \vec{c}$ are coplanar there must exist a relation of the form

$x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$, $\quad (x, y, z \in R$, not all zero) $\quad . . . (i)$

$\therefore \quad y\vec{b} + z\vec{c} = -x\vec{a}$

$\Rightarrow \quad \dfrac{y\vec{b} + z\vec{c}}{y + z} = -\dfrac{x}{y + z}\vec{a}$



Now $\dfrac{y\vec{b} + z\vec{c}}{y + z}$ is a point on the line BC.

Equation of $\overrightarrow{OA}$ is $\vec{r} = -t\vec{a}$

Thus $-\dfrac{x}{y + a}\vec{a}$ is a point of AD.

From (ii), $\overrightarrow{OD} = \dfrac{x}{y + z}\vec{a}$

$$\therefore \quad \frac{OD}{AD} = \frac{|\overrightarrow{OD}|}{|\overrightarrow{AD}|} = \left(-\frac{x}{y + z}\right) \times \left(-\frac{y + z}{x + y + z}\right) = \frac{x}{x + y + z}$$

Similarly, $\dfrac{OE}{BE} = \dfrac{y}{x + y + z}$ and $\dfrac{O}{CF} = \dfrac{z}{x + y + z}$

Adding, $\dfrac{OD}{AD} + \dfrac{OE}{BE} + \dfrac{OE}{CF} = \dfrac{x + y + z}{x + y + z} = 1$

Example 9 :

Find the ratio in which the yz–plane divides the line joining the points (3, 5, –7) and (–2, 1, 8). Find also the coordinates of the point of division.

Solution :

Let the yz–plane divide the line joining the given points in the ratio $m_1 : m_2$. Then the coordinates of the point of division are

$$\left(\frac{-2m_1 + 3m_2}{m_1 + m_2}, \frac{m_1 + 5m_2}{m_1 + m_2}, \frac{8m_1 - 7m_2}{m_1 + m_2}\right).$$

Since this point lies on the yz–plane, its x-coordinates is zero.
Therefore

$$-2m_1 + 3m_2 = 0, \text{ i.e. } m_1 : m_2 = 3 : 2$$

The other coordinates of the point of division are now

$$y = \frac{m_1 + 5m_2}{m_1 + m_2} = \frac{3 + 2.5}{3 + 5} = \frac{13}{5},$$

and $$z = \frac{8m_1 - 7m_2}{m_1 + m_2} = \frac{3.8 - 2.7}{3 + 2} = 2$$

Example 10 :

Prove that the three points P, Q, R whose coordinates are respectively (2, 5, –4), (1, 4, –3), (4, 7, –6) are collinear and find the ratio in which the point Q divides PR.

Solution :

We can prove that collinearity of the points P, Q, R by showing that PQ + PR = QR, so that the point P lies on (within) the segment of the line QR. Alternatively, we may proceed as follows.

Supposing that the points P, Q, R are collinear, let the point Q divide the line segment PR in the ratio $m_1 : m_2$. Then the coordinates of Q are

$$\left(\frac{4m_1+2m_2}{m_1+m_2}, \frac{7m_1+5m_2}{m_1+m_2}, \frac{-6m_1-4m_2}{m_1+m_2}\right)$$

We can find the ratio $m_1 : m_2$ by equating any one of these coordinates to the given coordinates of Q. Thus equating the x–coordinates, we get $4m_1 + 2m_2 = m_1 + m_2$, whence $m_1 : m_2 = -1 : 3$

It is necessary to verify that the same result is obtained by equating the other two coordinates. The only it will follow that the assumption of the collinearity of P, Q, R is correct.

Example 11 :

Show that the angles between the four diagonals of a rectangular parallelepiped whose edges are a, b, c are $\cos^{-1}\left(\frac{a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right)$.

Solution :

Let one vertex of the parallelepiped be taken as the origin O of coordinates, and the three edges OA, OB, OC meeting at O be taken as the coordinates axes. Then the points O, A, B, C are respectively (0, 0, 0), (a, 0, 0), (0, b, 0), (0, 0, c).

Also the opposite vertices O', A', B', C' are respectively the points

(a, b, c), (0, b, c), (a, 0, c), (a, b, 0)

Therefore the direction numbers of the diagonals OO' and AA' are a, b, c and a, –b, –c, and the angle between them is $\cos^{-1}\frac{a.a+b(-b)+c(-c)}{\sqrt{(a^2+b^2+c^2)}\sqrt{(a^2+b^2+c^2)}}$.

i.e. $\cos^{-1}\frac{a^2-b^2-c^2}{a^2+b^2+c^2}$

Similarly the angle between the diagonals OO' and BB' (direction numbers a, –b, c) is $\cos^{-1}\frac{a^2-b^2+c^2}{a^2+b^2+c^2}$ and the angle between OO' and CC' (direction numbers a, b, –c) is $\cos^{-1}\frac{a^2+b^2-c^2}{a^2+b^2+c^2}$.

The angles between the diagonals AA' and BB', AA' and CC', BB' and CC' are similarly found to be given by one of the above expressions. It follows that the angle between any two diagonals of the parallelepiped is given by one of the expressions $\cos^{-1}\frac{a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}$,where the ambiguous sings are not both positive.

Example 12 :

Prove that the three lines drawn from O with direction numbers 2, 1, 5; 2, –1, 1; 6, –4, 1; are coplanar.

Solution :

If the lines are coplanar, the normal to the plane containing them is at right angles to each of them. Let ℓ, m, n be the direction numbers of the normal to the plane containing the given lines. Then the condition of perpendicular dicularity of the normal and the three lines gives

$$2\ell + \text{m} + 5\text{n} = 0,\ 2\ell - \text{m} + \text{n} = 0 \text{ and } 6\ell - 4\text{m} + \text{n} = 0$$

From the first two equations, we easily have $\frac{\ell}{3} = \frac{\text{m}}{4} = \frac{\text{n}}{-2}$ and these values also satisfy the third equation. It follows that a line (with direction numbers 3, 4, –2) exists which is perpendicular to the given lines, which all pass through a common point (the point O in the case).

Hence the lines are coplanar.

[It should be noted that the mere existence of a line perpendicular to all the given lines is not sufficient to ensure that they are coplanar; it also be shown that the intersect in a common point.]

Example 13 :

Find the direction cosines ℓ, m, n, of two lines which are connected by the relations $\ell - 5\text{m} + 3\text{n} = 0$ and $7\ell^2 + 5\text{m}^2 - 3\text{n}^2 = 0$.

Solution :

To find the values of ℓ, m, n, from the given relations, we have to solve these equations

From the first, we have $\ell = 5\text{m} - 3\text{n}$.

Substituting this value in the second equation, we get

$$7(5\text{m} - 3\text{n})^2 + 5\text{m}^2 - 3\text{n}^2 = 0,$$

i.e. $30(2\text{m} - \text{n})(3\text{m} - 2\text{n}) = 0$, i.e. $2\text{m} = \text{n}$ and $3\text{m} = 2\text{n}$

$$\text{Therefore } \frac{\text{m}}{1} = \frac{\text{n}}{2} = \frac{5\text{m} - 3\text{n}}{5 - 2.3} = \frac{\ell}{-1} = \frac{\sqrt{(\ell^2 + \text{m}^2 + \text{n}^2)}}{\sqrt{(1+4+1)}} = \frac{1}{\sqrt{6}},$$

and

$$\frac{\text{m}}{2} = \frac{\text{n}}{3} = \frac{5\text{m} - 3\text{n}}{5.2 - 3.3} = \frac{\ell}{1} = \frac{1}{\sqrt{14}}, \text{ similarly,}$$

Thus the direction cosines of the lines are

$$\frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \text{ and } \frac{1}{\sqrt{14}}, \frac{2}{14}, \frac{3}{14}$$

SOLVED OBJECTIVE EXAMPLES

Example 1 :

If a plane meets the co-ordinate axes in A, B, C such that the centroid of the triangle is the point $(1, r, r^2)$, then equation of the plane is

(A) $x + ry + r^2z = 3r^2$ (B) $r^2x + ry + z = 3r^2$

(C) $x + ry + r^2z = 3$ (D) $r^2x + ry + z = 3$

Solution :

Let an equation of the required plane be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

This meets the coordinates axes in

A(a, 0, 0), B(0, b, 0) and C(0, 0, c).

So that the coordinates of the centroid of the triangle ABC are $\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right) = (1, r, r^2)$ (given)

$\Rightarrow$ $a = 3,\ b = 3r,\ c = 3r^2$

Hence the required equation of the plane is

$$\frac{x}{3}+\frac{y}{3r}=1 \text{ or } r^2x + ry + z = 3r^2$$

Hence (B) is the correct answer.

Example 2 :

The volume of the tetrahedron included between the plane $3x + 4y - 5z - 60 = 0$ and the coordinate planes is

(A) 60 (B) 600

(C) 720 (D) none of these.

Solution :

Equation of the given plane can be written as

$$\frac{x}{20}+\frac{y}{15}+\frac{z}{-12}=1$$

which meets the coordinates axes in points A(20, 0, 0), B(0, 15, 0) and C(0, 0, - 12) and the coordinates of the origin are (0, 0, 0).

$\therefore$ the volume of the tetrahedron OABC is

$$\frac{1}{6}\begin{vmatrix} 20 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & -12 \end{vmatrix} = \left|\frac{1}{6}\times 20\times 15\times(-12)\right| = 600$$

Hence (B) is the correct answer.

Example 3 :

A line segment has length 63 and direction ratios are 3, -2, 6. If the line makes an obtuse angle with x-axis, the components of the line vector are

(A) 27, - 18, 54 (B) - 27, 18, 54
(C) - 27, 18, -54 (D) 27, - 18, - 54

Solution :

Let the components of the line vector be a, b, c. Then

$$a^2 + b^2 + c^2 = (63)^2 \quad \ldots (i)$$

Also $\frac{a}{3} = \frac{b}{-2} = \frac{c}{6} = \lambda$ (say), then $a = 3\lambda$, $b = -2\lambda$ and $c = 6\lambda$

and from (i) we have $9\lambda^2 + 4\lambda^2 + 36\lambda^2 = (63)^2 \Rightarrow 49\lambda^2 = (63)^2$

$$\Rightarrow \quad \lambda = \pm\frac{63}{7} = \pm 9$$

Since $a = 3\lambda < 0$ as the line makes an obtuse angle with x-axis, $\lambda = -9$ and the required components are –27, 18, –54.

Hence (C) is the correct answer.

Example 4 :

Lines $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-k}$ and $\frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{1}$ are coplanar, if

(A) k = 0 (B) k = - 1
(C) k = 2 (D) k = 3

Solution :

The given lines are coplanar if

$$0 = \begin{vmatrix} 2-I & 3-4 & 4-5 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1-k \\ k & k+2 & 1+k \end{vmatrix}$$

or if $2(1 + k) - (k + 2)(1 - k) = 0$ or if $k^2 + 3k = 0$

or if k = 0, - 3.

Hence (A) is the correct answer.

Example 5 :

The Cartesian equation of the plane passing through the line of intersection of the planes $r.(2\hat{i} - 3\hat{j} + 4\hat{k}) = 1$ and $r.(\hat{i} - \hat{j}) + 4 = 0$ and perpendicular to the plane $r(2\hat{i} - \hat{j} + \hat{k}) + 8 = 0$ is

(A) 3x - 4y + 4z = 5 (B) x - 2y + 4z = 3
(C) 5x - 2y - 12z + 47 = 0 (D) 2x + 3y + 4 = 0

Solution :

Equation of any plane passing through the intersection of the planes $r.(2\hat{i} - 3\hat{j} + 4\hat{k}) = 1$ and $r.(\hat{i} - \hat{j}) + 4 = 0$ is

$$2x - 3y + 4z - 1 + \lambda(x - y + 4) = 0$$

or $(2 + \lambda)x - (3 + \lambda)y + 4z + 4\lambda - 1 = 0$

This plane is perpendicular to the plane $r.(2\hat{i}-\hat{j}+\hat{k})+8=0$ if

$$2(2+\lambda)+(3+\lambda)+4=0.$$

$$11+3\lambda=0 \quad \Rightarrow \quad \lambda=-\frac{11}{3}$$

Hence the required equation of the plane is

$$3(2x-3y+4z-1)-11(x-y+4)=0 \Rightarrow 5x-2y-12z+47=0.$$

Hence (C) is the correct answer.

Example 6 :

The image of the point P(1, 3, 4) in the plane $2x-y+z+3=0$ is

(A) (3, 5, –2) (B) (–3, 5, 2)

(C) (3, –5, 2) (D) (3, 5, 2)

Solution :

Let image of the point P(1, 3, 4) in the given plane be the point Q. The equation of the line through P and normal to the given plane is $\frac{x-1}{2}=\frac{y-3}{-1}=\frac{z-4}{1}$

Since this line passes through Q, so let the coordinates of Q be $(2r+1, -r+3, r+4)$.

The coordinates of the mid-point of PQ are $\left(r+1, -\frac{r}{2}+3, \frac{r}{2}+4\right)$.

This point lies on the given plane. Therefore, $r=-2$.

Hence (B) is the correct answer because the coordinates of Q are (–3, 5, 2).

Example 7 :

The coordinates of a point of the line $\frac{x-1}{2}=\frac{y+1}{-3}=z$ at a distance $4\sqrt{14}$ from the point (1, –1, 0) nearer the origin are

(A) (9, –13, 4) (B) $\left(8\sqrt{14}+1, -12\sqrt{14}-1, 4\sqrt{14}\right)$

(C) (–7, 11, –4) (D) $\left(-8\sqrt{14}+1, 12\sqrt{14}-1, -a4\sqrt{14}\right)$

Solution :

The coordinates of any point on the given line are $(2r+1, -3r-1, r)$

The distance of this point from the point (1, –1, 0) is given to be $4\sqrt{14}$

$\Rightarrow \quad (2r)^2+(-3r)^2+(r)^2=(4\sqrt{14})^2$

$\Rightarrow \quad 14r^2=16\times14= r=\pm4$

So the coordinates of the required point are (9, –13, 4) or (–7, 11, – 4)

Out of which nearer the origin is (–7, 11, – 4)

Hence (C) is correct answer.

Example 8 :

The equation of the plane containing the line $\frac{x-x_1}{\ell}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ is $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$, where

(A) $ax_1 + by_1 + cz_1 = 0$ (B) $a\ell + bm + cn = 0$

(C) $\frac{a}{\ell} = \frac{b}{m} = \frac{c}{n}$ (D) $\ell x_1 + my_1 + nz_1 = 0$

Solution :

If the given plane contains the given line, then normal to the plane must be perpendicular to the line and the condition for the same is $a\ell + bm + cn = 0$.

Hence (B) is correct answer.

Example 9 :

If the x-coordinate of a point P in the join of Q(2, 2, 1) and R(5, 1, –2) is 4, then its z -coordinate is

(A) 2 (B) 1

(C) – 1 (D) – 2

Solution :

Suppose P divides QR in the ratio $\lambda : 1$. Then coordinates of P are $\left(\frac{5\lambda + 2}{\lambda + 1}, \frac{\lambda + 2}{\lambda + 1}, \frac{-2\lambda + 1}{\lambda + 1}\right)$.

It is given that the x-coordinate of P is 4.

$\therefore \quad \frac{5\lambda + 2}{\lambda + 1} = 4 \Rightarrow \lambda = 2$

So, z-coordinate of P is $\frac{-2\lambda + 1}{\lambda + 1} = \frac{-4 + 1}{2 + 1} = -1$.

Hence (C) is correct answer.

Example 10 :

Ratio in which the xy–plane divides the join of (1, 2, 3) and (4, 2, 1) is

(A) 3 : 1 internally (B) 3 : 1 externally

(C) 1 : 2 internally (D) 2 : 1 externally

Solution :

Suppose xy–plane divides the join of (1, 2, 3) and (4, 2, 1) in the ratio $\lambda : 1$. Then the coordinates of the point of division are $\left(\frac{4\lambda + 1}{\lambda + 1}, \frac{2\lambda + 2}{\lambda + 1}, \frac{\lambda + 3}{\lambda + 1}\right)$. This point lies on xy–plane

So, z–coordinate of = 0 $\Rightarrow \frac{\lambda + 3}{\lambda + 1} = 0 \Rightarrow \lambda = -3$

Hence (B) is correct answer.

Example 11 :

A(3, 2, 0), B(5, 3, 2) and C(–9, 6, –3) are the vertices of a triangle ABC. If the bisector of $\angle ABC$ meets BC at D, then coordinates of D are

(A) $\left(\frac{19}{8}, \frac{57}{16}, \frac{17}{16}\right)$ (B) $\left(-\frac{19}{8}, \frac{57}{16}, \frac{17}{16}\right)$

(C) $\left(\frac{19}{8}, -\frac{57}{16}, \frac{17}{16}\right)$ (D) none of these

Solution :

D divides BC in the ratio AB : AC i.e. 3 : 13. Therefore coordinates of D are

$$\left(\frac{3\times -9+13\times 5}{3+13}, \frac{3\times 6+13\times 3}{3+13}, \frac{3\times -3+13\times 2}{3+13}\right) \quad \text{or} \quad \left(\frac{19}{8}, \frac{57}{16}, \frac{17}{16}\right)$$

Hence (A) is correct answer.

Example 12 :

A line passes through the points (6, –7, –1) and (2, –3, 1). The direction cosines of the line so directed that the angle made by it with the positive direction of x-axis is acute, are

(A) $\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}$ (B) $-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}$

(C) $\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}$ (D) $\frac{2}{3}, \frac{2}{3}, \frac{1}{3}$

Solution :

Let ℓ, m, n be the DC of the given the line. Then as it makes an acute angle with x-axis, therefore $\ell > 0$. The line passes through (6, –7, –1) and (2, –3, 1), therefore its DR are

6 – 2, – 7 + 3, –1 – 1 or 4, – 4, – 2 or 2, – 2, – 1

DC of the given line are $\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}$

Hence (A) is correct answer.

Example 13 :

If α, β, γ are the angle which a directed line makes with the positive directions of the coordinate axes, then $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ is equal to

(A) 1 (B) 2

(C) 3 (D) none of these

Solution :

The DC of the line are $\ell = \cos\alpha,\ m = \cos\beta,\ n = \cos\gamma$. Since $\ell^2 + m^2 + n^2 = 1$

$\therefore \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow \quad \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$ Hence (B) is correct answer.