SOLVED SUBJECTIVE EXAMPLES

***Example* 1 :**

If $A = \begin{bmatrix} 0 & -\tan\alpha/2 \\ \tan\alpha/2 & 0 \end{bmatrix}$ and I is a 2 × 2 unit matrix, then prove that

$$I + A = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \sin\alpha \end{bmatrix}$$

Solution :

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{and given } A = \begin{bmatrix} 0 & -\tan\alpha/2 \\ \tan\alpha/2 & 0 \end{bmatrix}$$

$$\therefore \quad I + A = \begin{bmatrix} 1 & -\tan\alpha/2 \\ \tan\alpha/2 & 1 \end{bmatrix} \qquad \text{... (1)}$$

$$\text{R.H.S.} = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \tan\alpha/2 \\ -\tan\alpha/2 & 1 \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \tan\alpha/2 \\ -\tan\alpha/2 & 1 \end{bmatrix}\begin{bmatrix} \dfrac{1-\tan^2\alpha/2}{1+\tan^2\alpha/2} & -\dfrac{2\tan\alpha/2}{1+\tan^2\alpha/2} \\ \dfrac{2\tan\alpha/2}{1+\tan^2\alpha/2} & \dfrac{1-\tan^2\alpha/2}{1+\tan^2\alpha/2} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1-\tan^2\alpha/2}{1+\tan^2\alpha/2} + \dfrac{2\tan^2\alpha/2}{1+\tan^2\alpha/2} & -\dfrac{2\tan\alpha/2}{1+\tan^2\alpha/2} + \dfrac{\tan\alpha/2(1-\tan^2\alpha/2)}{1+\tan^2\alpha/2} \\ \dfrac{-\tan\alpha/2(1-\tan^2\alpha/2}{1+\tan^2\alpha/2} + \dfrac{2\tan\alpha/2}{1+\tan^2\alpha/2} & \dfrac{2\tan^2\alpha/2}{1+\tan^2\alpha/2} + \dfrac{1-\tan^2\alpha/2}{1+\tan^2\alpha/2} \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{(1+\tan^2\alpha/2)}{(1+\tan^2\alpha/2} & \dfrac{-\tan\alpha/2\,(1+\tan^2\alpha/2)}{(1+\tan^2\alpha/2} \\ \dfrac{\tan\alpha/2\,(1+\tan^2\alpha/2)}{(1+\tan^2\alpha/2)} & \dfrac{(1+\tan^2\alpha/2)}{(1+\tan^2\alpha/2)} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\tan\alpha/2 \\ \tan\alpha/2 & 1 \end{bmatrix} = I + A = \text{L.H.S.} \qquad \{\text{from}\,(1)\}$$

***Example* 2 :**

If $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix}$, show that $A^3 = pI + qA + rA^2$

Solution :

We have $A^2 = AA$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix} \quad = \begin{bmatrix} 0 & 0 & 1 \\ p & q & r \\ pr & p+qr & q+r^2 \end{bmatrix}$$

$$\therefore \quad A^3 = A^2.A = \begin{bmatrix} 0 & 0 & 1 \\ p & q & r \\ pr & p+qr & q+r^2 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix}$$

$$= \begin{bmatrix} p & q & r \\ pr & p+qr & q+r^2 \\ pq+r^2p & pr+q^2+qr^2 & p+2qr+r^3 \end{bmatrix}$$

and $pI + qA + rA^2 = p\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + q\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix} + r\begin{bmatrix} 0 & 0 & 1 \\ p & q & r \\ pr & p+qr & q+r^2 \end{bmatrix}$

$$= \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix} + \begin{bmatrix} 0 & q & 0 \\ 0 & 0 & q \\ pq & q^2 & qr \end{bmatrix} + \begin{bmatrix} 0 & 0 & r \\ pr & qr & r^2 \\ pr^2 & pr+qr^2 & qr+r^3 \end{bmatrix}$$

$$= \begin{bmatrix} p+0+0 & 0+q+0 & 0+0+r \\ 0+0+pr & p+0+qr & 0+q+r^2 \\ 0+pq+pr^2 & 0+q^2+pr+qr^2 & p+qr+qr+r^3 \end{bmatrix}$$

$$= \begin{bmatrix} p & q & r \\ pr & p+qr & q+r^2 \\ pq+pr^2 & q^2+pr+qr^2 & p+2qr+r^3 \end{bmatrix}$$

Example 3 :

Verify that $A = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ -2 & -2 & -1 \end{pmatrix}$ is an orthogonal matrix.

Solution :

$$A = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ -2 & -2 & -1 \end{pmatrix}$$

$$\therefore \qquad A' = \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ 2 & 2 & -1 \end{pmatrix}$$

$$\therefore \qquad AA' = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ -2 & -2 & -1 \end{pmatrix} \times \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ 2 & 2 & -1 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 1+4+4 & -2-2+4 & -2+4-2 \\ -2-2+4 & 4+1+4 & 4-2-2 \\ -2+4-2 & 4-2-2 & 4+4+1 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Hence A is an orthogonal matrix.

Example 4 :

If A and B are two non-zero square matrices of the same order n such that AB = O, then $|A| = |B| = 0$. Further show that $|C| = 0 \Leftrightarrow |\text{adj } C| = 0$, for any square matrix C.

Solution :

If $|A| \neq 0$, then A^{-1} exists. Hence $AB = 0 \Rightarrow A^{-1}(AB) = A^{-1}O \Rightarrow B = O$, which is not the case. Hence $|A| = 0$. Similarly $|B| = 0$.

If $|C| = O$, then $C(\text{adj } C) = |C|\, I = 0 \Rightarrow |\text{adj } C| = 0$, as shown earlier conversely if $|\text{adj } C| = 0$ then $C(\text{adj } C) = |C|\, I \Rightarrow |C|.\ |\text{adj } C| = |C|^n \Rightarrow |C| = 0$.

Example 5 :

In the equations $x = cy + bz$, $y = az + cx$, $z = bx + ay$, where x, y, z are not all zero, prove that

(i) $a^2 + b^2 + c^2 + 2abc = 1$

(ii) $\frac{x^2}{1-a^2} = \frac{y^2}{1-b^2} = \frac{z^2}{1-c^2}$.

Solution :

(i) Since x, y and z are not all zero,

$\therefore$ the given equations

$x - cy - bz = 0 \qquad \ldots (1)$

$cx - y + az = 0$ . . . (2)

and $bx + ay - z = 0$ . . . (3)

have a nontrivial solution.

$$\therefore \quad \Delta = 0 \text{ i.e. } \begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0$$

or $1[1 - a^2] + c[- c - ab] - b[ca + b] = 0$ [expressing by R_1]

or $1 - a^2 - c^2 - abc - abc - b^2 = 0$ . . . (4)

or $a^2 + b^2 + c^2 + 2abc = 1$

(ii) By cross-multiplication (1) and (2), we have

$$\frac{x}{-ca-b} = -\frac{y}{-bc-a} = -\frac{z}{-1+c^2}$$

squaring $\dfrac{x^2}{c^2a^2+1-a^2-c^2} = \dfrac{y^2}{b^2c^2+1-b^2-c^2} = \dfrac{z^2}{\left(1-c^2\right)^2}$ [Using (4)]

or $$\frac{x^2}{\left(1-a^2\right)\left(1-c^2\right)} = \frac{y^2}{\left(1-b^2\right)\left(1-c^2\right)} = \frac{z^2}{\left(1-c^2\right)^2}$$

Hence $\dfrac{x^2}{1-a^2} = \dfrac{y^2}{1-b^2} = \dfrac{z^2}{1-c^2}$.

Example 6 :

Let $A = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$, where $a \neq 0$. Show that for $n \in N, A^n = \begin{bmatrix} a^n & \dfrac{b(a^n-1)}{(a-1)} \\ 0 & 1 \end{bmatrix}$

Solution :

We have to show by mathematical induction

Step I : For n = 1,

$$A^1 = \begin{bmatrix} a & b\dfrac{(a-1)}{(a-1)} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

Hence the result is true for n = 1

Step II: Assume the result to be true for some $k \geq 0$.

$$\therefore \qquad A^k = \begin{bmatrix} a^k & b\dfrac{(a^k-1)}{(a-1)} \\ 0 & 1 \end{bmatrix}$$

Step III: For n = k + 1

$$A^{k+1} = A^k.\,A \quad = \begin{bmatrix} a^k & \dfrac{b(a^k-1)}{(a-1)} \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a^k.a + 0 & a^k.b + \dfrac{b(a^k-1)}{(a-1)}.1 \\ 0+0 & 0+1.1 \end{bmatrix} = \begin{bmatrix} a^{k+1} & b\left\{\dfrac{a^{k+1}-1}{(a-1)}\right\} \\ 0 & 1 \end{bmatrix}$$

Hence the result is true for k + 1. Therefore by the principle of induction, the result is true for all $n \in N$.

Example 7 :

By the method of matrix inversion, solve the system.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x & u \\ y & v \\ z & w \end{bmatrix} = \begin{bmatrix} 9 & 12 \\ 52 & 15 \\ 0 & -1 \end{bmatrix}$$

Solution :

We have $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x & y \\ y & v \\ z & w \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & -1 \end{bmatrix}$

or $\quad AX = B \quad$ or $\quad X = A^{-1} B \qquad$ . . . (i)

Where $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix}, X = \begin{bmatrix} x & u \\ y & v \\ z & w \end{bmatrix}$ and $B = \begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & 9 \end{bmatrix}$

$|A| = 1\,(-5-7) - 1\,(-2-14) + 1\,(2-10) = -12 + 16 - 8 = -4 \neq 0$

Let C be the matrix of cofactors of elements of |A|.

$$\therefore \quad C=\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} \begin{vmatrix} 5 & 7 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 7 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 5 \\ 2 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 5 & 7 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 7 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -12 & 16 & -8 \\ 2 & -3 & 1 \\ 2 & -5 & 3 \end{bmatrix}$$

$$\therefore \quad \text{Adj } A = C' = \begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix} \quad \therefore \quad A^{-1} = \frac{\text{Adj.A}}{|A|} = -\frac{1}{4}\begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}$$

Now, $A^{-1}B = -\frac{1}{4}\begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix} \times \begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & -1 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} -4 & 4 \\ -12 & -8 \\ -20 & -4 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ 5 & 1 \end{bmatrix}$

from (1) $X = A^{-1}B \Rightarrow \begin{bmatrix} x & u \\ y & v \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ 5 & 1 \end{bmatrix}$

On equating the corresponding elements, we have

$x = 1, u = -1$
$y = 3, v = 2$
$z = 5, w = 1$

Example 8 :

If a, b and c are all different and if $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$, prove that $abc = -1$.

Solution :

$$D = \begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$$

$$= \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (-1)^1 \begin{vmatrix} 1 & a^2 & a \\ 1 & b^2 & b \\ 1 & c^2 & c \end{vmatrix} + abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} [C_1 \leftrightarrow C_3 \text{ in 1st det.}]$$

$$= (-1)^2 \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} [C_2 \leftrightarrow C_3 \text{ in 1st det.}]$$

Thus (abc + 1). $\Delta = 0 \Rightarrow abc = -1$, as $\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \neq 0$ (a, b and c are all different).

Example 9 :

For what value of k the following system of equations :

x + ky + 3z = 0

3x + ky - 2z = 0

2x + 3y – 4z = 0 possess a non-trivial solution. For that value of k, find all the solutions of the system.

Solution :

For the nontrivial solution

$$\therefore \quad \begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 3 & -4 \end{vmatrix} = 0 \quad \Rightarrow \quad k = 33/2$$

Putting the value of k in the given equations, the equations become

$$x + \frac{33}{2}y + 3z = 0 \qquad \text{... (i)}$$

$$3x + \frac{33}{2}y - 2z = 0 \qquad \text{... (ii)}$$

$$2x + 3y - 4z = 0 \qquad \text{... (iii)}$$

Multiply (i) by 3 and subtract from (ii) to get

$-33y - 11z = 0$

or $z = -3y$ ... (iv)

Now let y = λ, $\therefore$ $z = -3\lambda$

from (iii), $2x + 3\lambda + 12\lambda = 0$

$$\therefore \quad x = -\frac{15\lambda}{2}, \quad \lambda \in R$$

Example 10 :

Let $x_1 = 3y_1 + 2y_2 - y_3$, $y_1 = z_1 - z_2 + z_3$
$x_2 = -y_1 + 4y_2 + 5y_3$, $y_2 = z_2 + 3z_3$
$x_3 = y_1 - y_2 + 3y_3$, $y_3 = 2z_1 + z_2$.
Express x_1, x_2, x_3 in terms of z_1, z_2, z_3.

Solution :

We have $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \\ -1 & 4 & 5 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \\ -1 & 4 & 5 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 3 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

$$\Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 9 \\ 9 & 10 & 11 \\ 7 & 1 & -2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$\therefore \quad x_1 = z_1 - 2z_2 + 9z_3$, $x_2 = 9z_1 + 10z_2 + 11z_3$, $x_3 = 7z_1 + z_2 - 2z_3$

Example 11 :

If $\Delta(x) = \begin{vmatrix} \alpha+x & \theta+x & \lambda+x \\ \beta+x & \phi+x & \mu+x \\ \gamma+x & \psi+x & v+x \end{vmatrix}$, show that $\Delta''(x)=0$ and $\Delta(x)=\Delta(0)+Sx$, where S denote the sum of all the cofactors of all elements in $\Delta(0)$ and dash denotes the derivative with respect to x.

Solution :

$$\Delta'(x) = \begin{vmatrix} 1 & \theta+x & \lambda+x \\ 1 & \phi+x & \mu+x \\ 1 & \psi+x & v+x \end{vmatrix} + \begin{vmatrix} \alpha+x & 1 & \lambda+x \\ \beta+x & 1 & \mu+x \\ \gamma+x & 1 & v+x \end{vmatrix} + \begin{vmatrix} \alpha+x & \theta+x & 1 \\ \beta+x & \phi+x & 1 \\ v+x & \psi+x & 1 \end{vmatrix}$$

Applying $C_2 \to C_2 - xC_1$ and $C_3 \to C_3 - xC_1$ in first and $C_1 \to C_1 - xC_2, C_3 \to C_3 - xC_2$ in second and $C_1 \to C_1 - xC_3$ and $C_2 \to C_2 - xC_3$ in third to get

$$\Delta'(x) = \begin{vmatrix} 1 & \theta & \lambda \\ 1 & \phi & \mu \\ 1 & \psi & v \end{vmatrix} + \begin{vmatrix} \alpha & 1 & \lambda \\ \beta & 1 & \mu \\ \gamma & 1 & v \end{vmatrix} + \begin{vmatrix} \alpha & \theta & 1 \\ \beta & \phi & 1 \\ \gamma & \psi & 1 \end{vmatrix} \Rightarrow \Delta''(x) = 0 .$$

If S is the sum of all the cofactors of all elements in $\Delta(0)$, then it can be seen that $\Delta'(x) = S$.

on integrating $\Delta(x) = Sx + c$

$\Rightarrow \quad \Delta(0) = 0 + c$

Hence $\Delta(x) = Sx + \Delta(0)$.

***Example* 12 :**

If α, β are the roots of the equation $ax^2 + bx + c = 0$ and $s_n = \alpha^n + \beta^n$, evaluate

$\begin{vmatrix} 3 & 1+s_1 & 1+s_2 \\ 1+s_1 & 1+s_2 & 1+s_3 \\ 1-s_2 & 1+s_3 & 1+s_4 \end{vmatrix}$ in terms of a, b, and c only.

Solution :

$$\begin{vmatrix} 1+1+1 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \end{vmatrix}$$

$$= \left((1-\alpha)(\alpha-\beta)(\beta-1)\right)^2$$

$$= \left(1-(\alpha+\beta)+\alpha\beta\right)^2 (\alpha-\beta)^2$$

$$= \left(1+\frac{b}{a}+\frac{c}{a}\right)^2 \left[\frac{b^2}{a^2}-\frac{4c}{a}\right]$$

$$= \frac{(a+b+c)^2\left(b^2-4ac\right)}{a^4}.$$

***Example* 13 :**

Prove that

$$\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4(b + c)(c + a)(a + b)$$

Solution :

Let $\Delta = \begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix}$

Putting $a + b = 0$

$\Rightarrow \quad b = -a$

then $\Delta = \begin{vmatrix} -2a & 0 & a+c \\ 0 & 2a & c-a \\ c+a & c-a & -2c \end{vmatrix}$

Expanding along R_1

$= -2a\{-4ac - (c-a)^2\} - 0 + (a+c)\{0 - 2a(c+a)\}$

$= 2a(c+a)^2 - 2a(c+a)^2$

$= 0$

Hence a + b is a factor of Δ Similarly b + c and c + a are the factors of Δ.

On expansion of determinant we can see that each term of the determinant is a homogeneous expression in a, b, c of degree 3 and also R.H.S is a homogeneous expression of degree 3.

Let $\Delta = k(a+b)(b+c)(c+a)$

or $\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = k(a+b)(b+c)(c+a)$

If we choose a = 0, b = 1, c = 2, we get

$\Rightarrow \quad 0 - 1(-4-6) + 2(3+4) = 6k$

$\Rightarrow k = 4$

Hence $\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4(a+b)(b+c)(c+a)$

***Example* 14 :**

If f(x) is a polynomial of degree < 3, prove that

$$\begin{vmatrix} 1 & a & f(a)/(x-a) \\ 1 & b & f(b)/(x-b) \\ 1 & c & f(c)/(x-c) \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \frac{f(x)}{(x-a)(x-b)(x-c)}$$

Solution :

$$\frac{f(x)}{(x-a)(x-b)(x-c)} \equiv \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)} \qquad \text{... (1) (say)}$$

On comparing the various powers of x, we get

$$\begin{cases} A = -\dfrac{f(a)}{(a-b)(c-a)} \\ B = -\dfrac{f(b)}{(a-b)(b-c)} \\ C = -\dfrac{f(c)}{(b-c)(c-a)} \end{cases}$$

Now from (1) we have

$$\frac{f(x)}{(x-a)(x-b)(x-c)} = \frac{(c-b)\dfrac{f(a)}{(x-a)} + (a-c)\dfrac{f(b)}{x-b} + (b-a)\dfrac{f(c)}{(x-c)}}{(a-b)(b-c)(c-a)} = \frac{\begin{vmatrix} 1 & a & f(a)/(x-a) \\ 1 & b & f(b)/(x-b) \\ 1 & c & f(c)/(x-c) \end{vmatrix}}{\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}}$$

Example 15 :

If A, B and C are the angles of a triangle, show that

(i) $$\begin{vmatrix} \sin 2A & \sin C & \sin B \\ \sin C & \sin 2B & \sin A \\ \sin B & \sin A & \sin 2C \end{vmatrix} = 0$$

(ii) $$\begin{vmatrix} -1+\cos B & \cos C+\cos B & \cos B \\ \cos C+\cos A & -1+\cos A & \cos A \\ -1+\cos B & -1+\cos A & -1 \end{vmatrix} = 0$$

Solution :

(i) $$\text{L.H.S} = \begin{vmatrix} \sin 2A & \sin C & \sin B \\ \sin C & \sin 2B & \sin A \\ \sin B & \sin A & \sin 2C \end{vmatrix} = \begin{vmatrix} 2ka\cos A & kc & kb \\ kb & 2kb\cos B & ka \\ kb & ka & 2kc\cos C \end{vmatrix} \text{(from sine rule)}$$

$$= k^3 \begin{vmatrix} 2a\cos A & c & b \\ c & 2b\cos B & a \\ b & a & 2c\cos c \end{vmatrix}$$

$$= k^3 \begin{vmatrix} a\cos A + a\cos A & a\cos B + b\cos A & a\cos C + c\cos A \\ a\cos B + b\cos A & b\cos B + b\cos B & b\cos C + c\cos B \\ a\cos C + c\cos A & b\cos C + c\cos B & c\cos C + c\cos C \end{vmatrix}$$

$$= k^3 \begin{vmatrix} \cos A & a & 0 \\ \cos B & b & 0 \\ \cos C & c & 0 \end{vmatrix} \times \begin{vmatrix} a & \cos A & 0 \\ b & \cos B & 0 \\ c & \cos C & 0 \end{vmatrix} = 0 \times 0 = 0 = \text{R.H.S.}$$

(ii) L.H.S. = $\begin{vmatrix} -1+\cos B & \cos C + \cos B & \cos B \\ \cos C + \cos A & -1+\cos A & \cos A \\ -1+\cos B & -1+\cos A & -1 \end{vmatrix}$

Applying $C_1 \to C_1 - C_3 : C_2 \to C_2 - C_3$

$$= \begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix} = \frac{1}{a} \begin{vmatrix} -a & \cos C & \cos B \\ a\cos C & -1 & \cos A \\ a\cos B & \cos A & -1 \end{vmatrix}$$

Applying $C_1 \to C_1 + bC_2 + cC_3$

$$= \frac{1}{a} \begin{vmatrix} 0 & \cos C & \cos B \\ 0 & -1 & \cos A \\ 0 & \cos A & -1 \end{vmatrix} = 0 \quad \text{R.H.S.}$$

SOLVED OBJECTIVE EXAMPLES

***Example* 1 :**

If $\begin{bmatrix} x^2-4x & x^2 \\ x^2 & x^3 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -x+2 & 1 \end{bmatrix}$, then x =

(A) 1 (B) – 1

(C) – 2 (d) 3

Solution :

$x^2 - 4x = -3 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow x = 1, 3$

$x^2 = 1 \Rightarrow x = \pm 1$

$x^2 = -x + 2 \Rightarrow x^2 + x - 2 = 0 \Rightarrow x = -2, 1$

$x^3 = 1 \Rightarrow x = 1, \omega, \omega^2$

$\therefore$ Common value of x is 1.

Hence (A) is correct.

***Example* 2 :**

If the trace of the matrix $A = \begin{bmatrix} x-1 & 0 & 2 & 5 \\ 3 & x^2-2 & 4 & 1 \\ -1 & -2 & x-3 & 1 \\ 2 & 0 & 4 & x^2-6 \end{bmatrix}$ is 0, then x is equal to

(A) –2, 3 (B) 2, –3

(C) –3, 2 (d) 3, –2

Solution :

Trace of matrix is defined as $\sum_{i=1}^{n} a_{ii} = 2x^2 + 2x - 12 = 0$

$\Rightarrow x = -3, 2$

Hence (C) is correct

***Example* 3 :**

If A and B are square matrices of order 3, then

(A) adj (AB) = –adjA + adj B (B) $(A + B)^{-1} = A^{-1} + B^{-1}$

(C) $AB = 0 \Rightarrow |A| = 0$ or $|B| = 0$ (d) $AB = 0 \Rightarrow |A| = 0$ and $|B| = 0$

Solution :

If AB = 0, then at least one of A and B is necessarily singular.

Hence (C) is correct.

***Example* 4 :**

If A and B are any two square matrices of the same order, then

(A) $(AB)' = A'B'$ (B) adj(AB) = adj(A) adj(B)

(C) $(AB)' = B'A'$ (d) $AB = 0 \Rightarrow A = O$ or $B = O$

Solution :

It is a known fact that $(AB)' = B'A'$.

Hence (C) is the correct answer.

Example 5 :

Let A be a square matrix of order n & k be a scalar. Then |kA| equals

(A) k |A| (B) |k| |A|

(C) k^n |A| (D) none of these

Solution :

KA is the matrix, in which all the entries of A are multiplied by K.

Hence $|KA| = K^n |A|$, taking K common from all the columns.

Hence (C) is correct.

Example 6 :

If $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, then the value of $\left|A'\ A^{-1}\right|$ is

(A) cos4x (B) $\sec^2 x$

(C) –cos4x (D) 1

Solution :

$$A' = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{1+\tan^2 x}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}, \quad A'A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$$

$|A' A^{-1}| = 1$

Hence (d) is correct.

Example 7 :

The digits A, B and C are such that the three digit numbers A88, 6B8, 86C are divisible by 72 then

the determinant $\begin{vmatrix} A & 6 & 8 \\ 8 & B & 6 \\ 8 & 8 & C \end{vmatrix}$ is divisible by

(A) 72 (B) 144

(C) 288 (d) 216

Solution :

$$R_3 \to 100R_1 + 10R_2 + R_3 \Rightarrow \begin{vmatrix} A & 6 & 8 \\ 8 & B & 6 \\ 8 & 8 & C \end{vmatrix} = \begin{vmatrix} A & 6 & 8 \\ 8 & B & 6 \\ A88 & 6BC & 86C \end{vmatrix}$$

which is divisible by 72. Hence (A) is correct.

Example 8 :

Maximum value of $\begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1+\cos^2 x & \sin 2x \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix}$ is

(A) 4
(B) 6
(C) 2
(d) none of these

Solution :

Applying $C_1 \to C_1 + C_2$ we get

$$\Delta = \begin{vmatrix} 2 & \cos^2 x & 4\sin 2x \\ 2 & 1+\cos^2 x & 4\sin 2x \\ 1 & \cos^2 x & 1+4\sin 2x \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ we get ,

$$\Delta = \begin{vmatrix} 2 & \cos^2 x & 4\sin 2x \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{vmatrix} = 2 + 4\sin 2x \le 6$$

Hence maximum value is 6 and (B) is the correct answer.

Example 9 :

If $\Delta_r = \begin{vmatrix} 2r-1 & {}^mC_r & 1 \\ m^2-1 & 2^m & m+1 \\ \sin^2(m^2) & \sin^2(m) & \sin^2(m+1) \end{vmatrix}$ then $\sum_{r=0}^{m} \Delta_r$ is equal to

(A) $m^2 - 1$
(B) 2^m
(C) zero
(d) none of these

Solution :

$$\Delta_r = \begin{vmatrix} 2r-1 & {}^mC_r & 1 \\ m^2-1 & 2^m & m+1 \\ \sin^2(m^2) & \sin^2(m) & \sin^2(m+1) \end{vmatrix}$$

$$\Rightarrow \quad \sum_{r=0}^{m} \Delta_r = \begin{vmatrix} \sum_{r=0}^{m}(2r-1) & \sum_{r=0}^{m} {}^mC_r & \sum_{r=0}^{m} 1 \\ m^2-1 & 2^m & m+1 \\ \sin^2(m^2) & \sin^2(m) & \sin^2(m+1) \end{vmatrix} = \begin{vmatrix} m^2-1 & 2^m & m+1 \\ m^2-1 & 2^m & m+1 \\ \sin^2(m^2) & \sin^2(m) & \sin^2(m+1) \end{vmatrix} = 0$$

Hence (C) is correct.

Example 10 :

The system of linear equations $x + y + z = 2$, $2x + y - z = 3$, $3x + 2y + kz = 4$ has a unique solution if

(A) $k \neq 0$ (B) $-1 < k < 1$

(C) $-2 < k < 2$ (d) $k = 0$

Solution :

The system of equations has a unique solution if

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & k \end{vmatrix} \neq 0 \Rightarrow k \neq 0$$

Hence (A) is the correct answer.

Example 11 :

If A, B and C are the angles of a non-right angled triangle ABC, then the value of

$$\begin{vmatrix} \tan A & 1 & 1 \\ 1 & \tan B & 1 \\ 1 & 1 & \tan C \end{vmatrix} \text{ is equal to}$$

(A) 1 (B) 2

(C) –1 (d) –2

Solution :

Given determinant is equal to; $\tan A (\tan B.\ \tan C - 1) - 1 (\tan C - 1) + 1 (1 - \tan B)$

$= \tan A.\ \tan B.\ \tan C - \tan A - \tan B - \tan C + 2 = 2$(as $\prod \tan A = \Sigma \tan A$)

Hence (B) is correct.

Example 12 :

If x, y, z are non zero real numbers, then the values of $\begin{vmatrix} \frac{1}{x} & x^2 & yz \\ \frac{1}{y} & y^2 & zx \\ \frac{1}{z} & z^2 & xy \end{vmatrix}$ depends upon

(A) x only (B) y only

(C) z only (d) none of these

Solution :

Multiplication of R_1 by x, R_2 by y and R_3 by z, reduces the given determinant to,

$$\frac{1}{xyz}\begin{vmatrix} 1 & x^3 & xyz \\ 1 & y^3 & xyz \\ 1 & z^3 & xyz \end{vmatrix} = \begin{vmatrix} 1 & x^3 & 1 \\ 1 & y^3 & 1 \\ 1 & z^3 & 1 \end{vmatrix} = 0$$

Hence (d) is the correct answer.

***Example* 13 :**

Let f(x) = $\begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$. The value of $\lim_{x\to 0}\frac{f(x)}{x}$ is equal to

(A) 1 (B) –1

(C) 0 (d) none of these

Solution :

$$\lim_{x\to 0}\frac{f(x)}{x} = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = f'(0)$$

$$f'(x)=\begin{vmatrix} -\sin x & 1 & 0 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix} + \begin{vmatrix} \cos x & x & 1 \\ 2\cos x & 2x & 2 \\ \tan x & x & 1 \end{vmatrix} + \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \sin^2 x & 1 & 0 \end{vmatrix}$$

Thus $\lim_{x\to 0}\frac{f(x)}{x} = f'(0)=\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \sin^2 x & 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 0$

Hence (C) is correct.

***Example* 14 :**

If $a\neq p,\ b\neq q,\ c\neq r$ and $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix}=0$ then the value of $\frac{p}{p-a}+\frac{q}{q-b}+\frac{r}{r-c}$ is equal to

(A) –1 (B) 1

(C) –2 (d) 2

Solution :

$R_1 \to R_1 - R_2,\ R_2 \to -R_3$ reduces the determinant to,

$$\begin{vmatrix} p-a & b-q & 0 \\ 0 & q-b & c-r \\ a & b & r \end{vmatrix}=0$$

⇒ (p – a) (q – b) r + (r – c)b) + (q – b) ((r – c)a) = 0

⇒ (p – a)((q – b)(r - c) – (r–c)a) = 0

⇒ (p – a)((q – b)(r–c + c) – – c)(q – b– q)) – (q – b) (r – c) (p – a – p) = 0

⇒ (p – a) (q – b) (r – c) + c(p – a) (q – b) – (p – a) q – b) (r – c) + q(p – a) (r – c)
–(p – a) (q – b) (r – c) + p(q – b) (r – c) = 0

Dividing through out by (p – a) (q – b) (r – c) we get,

$$\frac{c}{r-c}+\frac{q}{q-b}+\frac{p}{p-a}=1$$

$$\Rightarrow \quad 1+\frac{c}{r-c}+\frac{q}{q-b}+\frac{p}{p-a}=2 \quad \Rightarrow \quad \frac{r}{r-c}+\frac{q}{q-b}+\frac{p}{p-a}=2$$

Hence (d) is correct.

Example 15 :

Let a, b and c be positive real numbers. The following system of equations in x, y and z $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,\ \frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\ \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ has

(A) no solution (B) unique solution

(C) infinitely many solutions (d) finitely many solutions

Solution :

Let $\frac{x^2}{a^2}=X,\ \frac{y^2}{b^2}=Y$ and $\frac{z^2}{c^2}=Z$, then the given system of equations is

X + Y – Z = 1, X – Y + Z = 1, – X + Y + Z = 1

The coefficient of matrix $A=\begin{bmatrix}1 & 1 & -1\\ 1 & -1 & 1\\ -1 & 1 & 1\end{bmatrix} \Rightarrow |A| \neq 0$

On solving, we get X = Y = Z = 1

Hence x = $\pm$ a, y = $\pm$ b, z = $\pm$ c

Thus (d) is the correct answer.